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CHARACTERIZING STABLE COMPLETE ERDŐS SPACE

BY

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ABSTRACT

We focus on the space $\mathfrak{E}_c^\omega$, the countable infinite power of complete Erdős space $\mathfrak{E}_c$. Both spaces are universal spaces for the class of almost zero-dimensional spaces. We prove that $\mathfrak{E}_c^\omega$ has the property that it is stable under multiplication with any complete almost zero-dimensional space. We obtain this result as a corollary to topological characterization theorems that we develop for $\mathfrak{E}_c^\omega$. We also show that σ-compacta are negligible in $\mathfrak{E}_c^\omega$ and that the space is countable dense homogeneous.

1. Introduction

All topological spaces are assumed to be separable and metrizable. An element X of a class of topological spaces is called the **stable space** for that class if for every nonempty Y in the class we have that $X \times Y$ is homeomorphic to X. Note that a stable space for a given class, if it exists, is topologically unique. Important examples of stable spaces are the Cantor set 2^ω for the zero-dimensional compacta, the space of irrational numbers $\mathbb{P}$ for the complete zero-dimensional spaces, the Hilbert cube Q for the compact absolute retracts, and Hilbert space ℓ^2 for the complete absolute retracts; see [4], [2], [22, §7.8], and [26].

In [18] Paul Erdős introduced the space $\mathfrak{E}_c$ which consists of all vectors in the real Hilbert space ℓ^2 all of whose coordinates are elements of the convergent sequence $\{0\} \cup \{1/n : n \in \mathbb{N}\}$. He proved that this space is totally disconnected,

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homeomorphic to its own square, but one-dimensional. Note that $\mathfrak{E}_{\mathrm{c}}$ is a closed subspace of ℓ^2 and thus complete. This space is now known as **complete Erdős space**.

Dijkstra, van Mill and Steprāns [15] proved that $\mathfrak{E}_{\mathrm{c}}$ is not homeomorphic to its own countably infinite Cartesian power $\mathfrak{E}_{\mathrm{c}}^{\omega}$. This result produced solutions to a series of problems in the literature; see [15] and [11]. Both $\mathfrak{E}_{\mathrm{c}}$ and $\mathfrak{E}_{\mathrm{c}}^{\omega}$ are universal elements of the class of almost zero-dimensional spaces. A space X is called **almost zero-dimensional** if every point $x \in X$ has arbitrarily small neighbourhoods U that can be written as an intersection of clopen subsets of the space. It was observed in [15] that $\mathfrak{E}_{\mathrm{c}}^{\omega}$ is "more universal" than $\mathfrak{E}_{\mathrm{c}}$ in the sense that every complete almost zero-dimensional space admits a *closed* imbedding in $\mathfrak{E}_{\mathrm{c}}^{\omega}$ but that $\mathfrak{E}_{\mathrm{c}}$ does not contain a closed copy of $\mathfrak{E}_{\mathrm{c}}^{\omega}$. This prompted Dijkstra, van Mill and Steprāns to speculate that $\mathfrak{E}_{\mathrm{c}}^{\omega}$ is the "maximal" element of the class of almost zero-dimensional spaces. We provide conclusive evidence in support of this idea by proving that $\mathfrak{E}_{\mathrm{c}}^{\omega}$ is the stable space for the complete almost zero-dimensional spaces. We therefore call $\mathfrak{E}_{\mathrm{c}}^{\omega}$ **stable complete Erdős space**.

We prove the stability of $\mathfrak{E}_{\mathrm{c}}^{\omega}$ by finding topological characterizations of the space. In §4 we present an extrinsic characterization, by which we mean a characterization that depends on a particular imbedding of the space in a space with more structure, in our case the graph of a certain vector valued function, called an ω-Lelek function. From there we proceed by finding intrinsic characterizations in §5, namely characterizations in terms of purely topological concepts that are internal to the space. Finally, in §6 we prove the stability theorem and we also prove that σ-compacta are negligible in $\mathfrak{E}_{\mathrm{c}}^{\omega}$ and that the space is countable dense homogeneous.

Dijkstra, van Mill and Steprāns [15] proved that the autohomeomorphism groups of the universal Menger continua and the Sierpiński carpet are not homeomorphic to $\mathfrak{E}_{\mathrm{c}}$, by showing that these spaces share certain topological properties with $\mathfrak{E}_{\mathrm{c}}^{\omega}$ which $\mathfrak{E}_{\mathrm{c}}$ lacks. The main conjecture is that all these homeomorphism groups are homeomorphic to stable complete Erdős space. We like to think that our characterization theorems for $\mathfrak{E}_{\mathrm{c}}^{\omega}$ are a step towards a possible proof of this conjecture.

2. Definitions and preliminaries

Let ω be the ordinal $\{0, 1, \dots\}$ as usual. We let $\mathbb{I}$ denote the closed interval $[0, 1]$.

Definition 2.1: A subset A of a space X is called a **C-set in** X if A can be written as an intersection of clopen subsets of X. A space is called **totally disconnected** if every singleton is a C-set. A space is called **almost zero-dimensional** if every point of the space has a neighbourhood basis consisting of C-sets of the space. If Z is a set that contains X, then we say that a (separable metric) topology $\mathcal{T}$ on Z **witnesses the almost zero-dimensionality of** X if $\dim(Z, \mathcal{T}) \leq 0$, $O \cap X$ is open in X for each $O \in \mathcal{T}$, and every point of X has a neighbourhood basis in X consisting of sets that are closed in $(Z, \mathcal{T})$. We will also say that the space $(Z, \mathcal{T})$ is a witness to the almost zero-dimensionality of X.

Every zero-dimensional space is almost zero-dimensional and every almost zero-dimensional space is totally disconnected. Almost zero-dimensionality is clearly hereditary and preserved under products. Oversteegen and Tymchatyn [23] have shown that every almost zero-dimensional space is at most one-dimensional; see also [21, 1].

Remark 2.2: Clearly, a space X is almost zero-dimensional if and only if there is a topology on X witnessing this fact. Let Z be a witness to the almost zero-dimensionality of some space X and let O be open in X. Then since X is separable metric we can write O as a union of countably many sets that are closed in Z. So every open subset of X is F_σ in Z and every closed subset of X is G_δ in X with respect to the witness topology.

A function $\varphi\colon X \to [-\infty, \infty]$ is called **upper semi-continuous** (**USC**) if $\{x \in X : \varphi(x) < t\}$ is open in X for every $t \in \mathbb{R}$. If $\varphi\colon X \to [0, \infty)$ is USC then we define

$$G_0^\varphi = \{(x, \varphi(x))\colon x \in X \text{ and } 0 < \varphi(x)\} \tag{2.1}$$

with the topology that is inherited from $X \times \mathbb{R}$. According to [12, Lemma 4.11] we have the following connection between witness topologies and USC functions; see also Abry and Dijkstra [1, Corollary 3].

LEMMA 2.3: *Let X be a space and let Z be a zero-dimensional space that contains X as a subset (but not necessarily as a subspace). Then the following statements are equivalent:*

(1) *Z is a witness to the almost zero-dimensionality of X and*
(2) *there exists a USC function $\varphi\colon Z \to \mathbb{I}$ such that the map $h\colon X \to G_0^\varphi$ that is defined by the rule $h(x) = (x, \varphi(x))$ is a homeomorphism.*

Definition 2.4: Let $\varphi\colon X \to [0,\infty]$ be a function and let X be a subset of a metric space (Y, d). We define $\operatorname{ext}_Y \varphi\colon Y \to [0,\infty]$ by

$$(\operatorname{ext}_Y \varphi)(y) = \lim_{\varepsilon \searrow 0}(\sup\{\varphi(z) : z \in X \text{ with } d(z,y) < \varepsilon\}) \quad \text{for } y \in Y, \tag{2.2}$$

where we use the convention $\sup \emptyset = 0$.

Note that the metric on Y is mentioned strictly for the sake of convenience and that the definition of $\operatorname{ext}_Y \varphi$ does not depend on the choice of d. It is easily seen that $\operatorname{ext}_Y \varphi$ is always USC, that it extends φ whenever φ is USC, and that the graph of φ is dense in the graph of $\operatorname{ext}_Y \varphi$ whenever X is dense in Y.

Consider now the real Hilbert space ℓ^2. This space consists of all sequences $z = (z_0, z_1, z_2, \dots) \in \mathbb{R}^\omega$ such that $\sum_{i=0}^\infty z_i^2 < \infty$. The topology on ℓ^2 is generated by the norm $\|z\| = (\sum_{i=0}^\infty z_i^2)^{1/2}$. The following two spaces were introduced by Erdős [18]: **Erdős space**

$$\mathfrak{E} = \{x \in \ell^2 \colon x_i \in \mathbb{Q} \text{ for each } i \in \omega\} \tag{2.3}$$

and **complete Erdős space**

$$\mathfrak{E}_\mathrm{c} = \{x \in \ell^2 \colon x_i \in \{0\} \cup \{1/n : n \in \mathbb{N}\} \text{ for each } i \in \omega\}. \tag{2.4}$$

Erdős showed that these spaces are one-dimensional, totally disconnected, and homeomorphic to their own square. The topologies of $\mathfrak{E}$ and $\mathfrak{E}_\mathrm{c}$ are characterized in [10, 12] and [13], respectively. Both spaces are universal elements of the class of almost zero-dimensional spaces; see [12, Theorem 4.15]. It is proved in [12, Corollary 9.4] that $\mathfrak{E}$ is homeomorphic to $\mathfrak{E}^\omega$.

If $0 < n < \omega$ then consider $[-\infty,\infty]^n = \prod_{i=0}^{n-1}[-\infty,\infty]$. We will represent elements $\mathbf{t} \in [-\infty,\infty]^n$ as follows: $\mathbf{t} = (t_0, t_1, \dots, t_{n-1}) = (t_i)_{i<n}$. We put $\mathbf{0} = (0,\dots,0) \in \mathbb{R}^n$ and $\mathbf{1} = (1,\dots,1) \in \mathbb{R}^n$. We also denote $[-\infty,\infty]^\omega = \prod_{i\in\omega}[-\infty,\infty]$ with analogous conventions. If $0 < n < \omega$ then we let the projection $\xi_n\colon [-\infty,\infty]^\omega \to [-\infty,\infty]^n$ be given by $\xi_n(\mathbf{t}) = (t_0, \dots, t_{n-1})$.

Now let $0 < \nu \le \omega$. We define a partial order $\le$ on $[-\infty,\infty]^\nu$ by $\mathbf{r} \le \mathbf{t}$ if $r_i \le t_i$ for every $i < \nu$. If $r_i < t_i$ for every $i < \nu$, then we put $\mathbf{r} \ll \mathbf{t}$. We extend the arithmetic operations on $\mathbb{R}$ over $\mathbb{R}^\nu$ by coordinate-wise evaluation. Thus, for instance, $\mathbf{r} + \mathbf{t} = (r_i + t_i)_{i<\nu}$. Similarly we define $\log\colon [0,\infty]^\nu \to [-\infty,\infty]^\nu$ by $\log \mathbf{t} = (\log t_i)_{i<\nu}$, where we extended the logarithm with $\log 0 = -\infty$ and $\log\infty = \infty$. We will also use the absolute value $|\mathbf{t}| = (|t_i|)_{i<\nu}$.

Let $\boldsymbol{\varphi} = (\varphi_i)_{i<\nu}$ and $\boldsymbol{\psi} = (\psi_i)_{i<\nu}$ be functions from spaces X respectively Y to $[-\infty,\infty]^\nu$. A pair $(h,\boldsymbol{\beta})$ is called a **homeomorphism from $\boldsymbol{\varphi}$ to $\boldsymbol{\psi}$** if $h\colon X \to Y$ is a homeomorphism and $\boldsymbol{\beta}\colon X \to (0,\infty)^\nu$ is continuous such that $\boldsymbol{\psi} \circ h = \boldsymbol{\beta}\cdot\boldsymbol{\varphi}$, where $\cdot$ stands for coordinate-wise multiplication. Being homeomorphic is obviously an equivalence relation for functions. We call $\boldsymbol{\varphi}$ **ν-USC** if every φ_i is USC. We define

$$M(\boldsymbol{\varphi}) = \left(\sup_{x\in X} |\varphi_i(x)|\right)_{i<\nu} \in [0,\infty]^\nu. \tag{2.5}$$

We will also use the notations

$$G^{\boldsymbol{\varphi}}_{\mathbf{0}} = \{(x,\boldsymbol{\varphi}(x))\colon x \in X \text{ and } \mathbf{0} \ll \boldsymbol{\varphi}(x)\} \tag{2.6}$$

and

$$L^{\boldsymbol{\varphi}}_{\mathbf{0}} = \{(x,\mathbf{t})\colon x \in X,\, \mathbf{t} \in [0,\infty]^\nu, \text{ and } \mathbf{t} \le \boldsymbol{\varphi}(x)\} \tag{2.7}$$

both equipped with the topology inherited from $X \times [0,\infty]^\nu$.

Remark 2.5: If $\varphi : X \to [0,\infty)$ is USC, then clearly L^φ_0 is a closed subset of $X \times \mathbb{R}$. In addition, $G^\varphi_0 = (X \times (0,\infty)) \cap L^\varphi_0 \setminus \bigcup_{i=1}^\infty L^{\varphi-1/i}_0$ is a G_δ-subset of $X \times \mathbb{R}$. These observations extend obviously to ν-USC functions. It is also clear that if $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ are homeomorphic, then $G^{\boldsymbol{\varphi}}_{\mathbf{0}}$ is homeomorphic to $G^{\boldsymbol{\psi}}_{\mathbf{0}}$.

Definition 2.6: Let $0 < \nu \le \omega$ and let $\boldsymbol{\varphi} : X \to [0,\infty)^\nu$ be ν-USC with $\dim X = 0$ (and hence $X \ne \emptyset$). We call $\boldsymbol{\varphi}$ a **ν-Lelek function** if $G^{\boldsymbol{\varphi}}_{\mathbf{0}}$ is dense in $L^{\boldsymbol{\varphi}}_{\mathbf{0}}$. A 1-Lelek function is simply called a **Lelek function**. If $0 < n < \omega$, then $\boldsymbol{\varphi}$ is a **greatest n-Lelek function** if φ is an n-Lelek function such that for every open cover $\mathcal{U}$ of X there exists a refinement $\mathcal{P}$ that is a clopen partition of X with the property that for each $P \in \mathcal{P}$ the restriction $\boldsymbol{\varphi}\restriction P$ assumes a greatest value, that is, there is a $p \in P$ with $\boldsymbol{\varphi}(p) = M(\boldsymbol{\varphi}\restriction P)$. An ω-Lelek function $\boldsymbol{\psi}$ is called a greatest ω-Lelek function if $\xi_n \circ \boldsymbol{\psi}$ is a greatest n-Lelek function for each $n \in \mathbb{N}$.

Remark 2.7: The following facts can be found in Lelek [20]. Lelek functions with compact domain C exist and C must be homeomorphic to the Cantor set. If φ is a Lelek function with a compactum C as domain and we identify the set $C \times \{0\}$ to a point in L_0^φ, then we obtain a **Lelek fan**. The end-point set of a Lelek fan G_0^φ is one-dimensional and topologically complete.

According to Bula and Oversteegen [5] and Charatonik [6] Lelek fans (and their end-point sets) are topologically unique.

Remark 2.8: Let φ be a Lelek function with a Cantor set C as domain. According to Kawamura, Oversteegen and Tymchatyn [19] complete Erdős space is homeomorphic to G_0^φ; see also Dijkstra [7]. Define the function $\boldsymbol{\psi}\colon C^\omega \to [0,\infty)^\omega$ by the rule $\boldsymbol{\psi}(x_0, x_1, \dots) = (\varphi(x_0), \varphi(x_1), \dots)$. It is easily verified that $\boldsymbol{\psi}$ is a greatest ω-Lelek function. Note that $\mathfrak{E}_{\mathrm{c}}^\omega$ is homeomorphic to $G_{\mathbf{0}}^{\boldsymbol{\psi}}$. This is our *standard model* for $\mathfrak{E}_{\mathrm{c}}^\omega$.

Definition 2.9: Let X be a space and let $\mathcal{A}$ be a collection of subsets of X. The space X is called **$\mathcal{A}$-cohesive** if every point of the space has a neighbourhood that does not contain nonempty clopen subsets of any element of $\mathcal{A}$. If a space X is $\{X\}$-cohesive, then we simply call X **cohesive**.

The standard examples of cohesive almost zero-dimensional spaces are $\mathfrak{E}$ and $\mathfrak{E}_{\mathrm{c}}$. This follows from Erdős' proof [18] that these spaces have the property that every clopen nonempty subset has diameter at least 1/2 with respect to the Hilbert norm; see also Dijkstra [7]. A cohesive space is obviously at least one-dimensional at every point but the converse is not valid; see Dijkstra [8]. We will use the following result from Dijkstra and van Mill [12, Lemma 5.9] about the connection between cohesion and Lelek functions.

LEMMA 2.10: *Let φ be a USC function from a zero-dimensional space X to $[0,\infty)$ and let $\mathcal{A}$ be a collection of subsets of X such that $\emptyset \notin \mathcal{A}$, G_0^φ is $\{G_0^{\varphi\restriction A}\colon A \in \mathcal{A}\}$-cohesive, and $\{x \in A\colon \varphi(x) > 0\}$ is dense in A for each $A \in \mathcal{A}$. Then there exists a USC function $\chi\colon X \to [0,\infty)$ such that $\chi \le \varphi$, the natural bijection h from the graph of φ to the graph of χ is continuous, the restriction $h\restriction G_0^\varphi\colon G_0^\varphi \to G_0^\chi$ is a homeomorphism, and for every $A \in \mathcal{A}$ we have that $\chi\restriction A$ is a Lelek function.*

3. Basic properties of ν-Lelek functions

Remark 3.1: Note that if $\boldsymbol{\varphi}$ is ν-Lelek, then $M(\boldsymbol{\varphi}) \gg \mathbf{0}$ and the domain of $\boldsymbol{\varphi}$ has no isolated points. The restriction of a (greatest) ν-Lelek function to a clopen nonempty set is clearly also a (greatest) ν-Lelek function. It is easily verified that every function that is homeomorphic to a ν-Lelek function is also a ν-Lelek function. We do not have a similar statement for greatest ν-Lelek functions. A function $\boldsymbol{\beta} : X \to \mathbb{R}^\nu$ is called a **step function** if every fibre is clopen in X. If $(h, \boldsymbol{\beta})$ is a homeomorphism from a greatest ν-Lelek function $\boldsymbol{\varphi}$ to a function $\boldsymbol{\psi}$ such that $\boldsymbol{\beta}$ is a step function, then $\boldsymbol{\psi}$ is also a greatest ν-Lelek function.

LEMMA 3.2: *If $\boldsymbol{\varphi}\colon X \to [0,\infty)^\omega$ is a function with a complete domain such that for each $n \in \mathbb{N}$, $\xi_n \circ \boldsymbol{\varphi}$ is n-Lelek, then $\boldsymbol{\varphi}$ is ω-Lelek.*

Proof. It suffices to show that $G_{\mathbf{0}}^{\boldsymbol{\varphi}}$ is dense in $L_{\mathbf{0}}^{\boldsymbol{\varphi}}$. Note that $L_{\mathbf{0}}^{\boldsymbol{\varphi}}$ is closed in $X \times \mathbb{R}^\omega$, thus it is a complete space. Define for $i \in \omega$ the set $A_i = \{(x, \boldsymbol{\varphi}(x)) : x \in X \text{ and } \varphi_i(x) > 0\}$ and note that it is a G_δ-subset of $L_{\mathbf{0}}^{\boldsymbol{\varphi}}$ by Remark 2.5. It is clear that A_i is dense in $L_{\mathbf{0}}^{\boldsymbol{\varphi}}$. Thus according to Baire $G_{\mathbf{0}}^{\boldsymbol{\varphi}} = \bigcap_{i=0}^\infty A_i$ is dense in $L_{\mathbf{0}}^{\boldsymbol{\varphi}}$. ∎

Definition 3.3: If $1 \le \nu \le \omega$, $\boldsymbol{\varphi}\colon X \to [0,\infty)^\nu$, and $\boldsymbol{\psi}\colon Y \to [0,\infty)^\nu$, then $\boldsymbol{\varphi} \times \boldsymbol{\psi}\colon X \times Y \to [0,\infty)^\nu$ is defined by $(\boldsymbol{\varphi} \times \boldsymbol{\psi})_i(x,y) = \varphi_i(x)\psi_i(y)$ for each $i < \nu$.

LEMMA 3.4: *If $1 \le \nu \le \omega$ and $\boldsymbol{\varphi}\colon X \to [0,\infty)^\nu$ and $\boldsymbol{\psi}\colon Y \to [0,\infty)^\nu$ are ν-USC functions, then $\boldsymbol{\varphi} \times \boldsymbol{\psi}$ is ν-USC as well and the natural map $h\colon G_{\mathbf{0}}^{\boldsymbol{\varphi}} \times G_{\mathbf{0}}^{\boldsymbol{\psi}} \to G_{\mathbf{0}}^{\boldsymbol{\varphi}\times\boldsymbol{\psi}}$ is a homeomorphism. If, moreover, $\boldsymbol{\varphi}$ is a ν-Lelek function and $G_{\mathbf{0}}^{\boldsymbol{\psi}}$ is dense in the graph of $\boldsymbol{\psi}$, then $\boldsymbol{\varphi} \times \boldsymbol{\psi}$ is a ν-Lelek function as well.*

Proof. The first statement follows immediately from the case $\nu = 1$ which was proved in [12, Lemma 4.14]. We verify the part about Lelek functions for the case $\nu = \omega$.

Let $(x,y) \in X \times Y$, let $\mathbf{t} \in [0,\infty)^\omega$ be such that $\mathbf{t} \le \boldsymbol{\varphi}(x)\boldsymbol{\psi}(y)$, let $U \times V$ be a neighbourhood of (x,y) in $X \times Y$, let $\varepsilon > 0$, and let $n \in \mathbb{N}$. Since $G_{\mathbf{0}}^{\boldsymbol{\psi}}$ is dense in the graph of $\boldsymbol{\psi}$ we can find a $y' \in V$ such that $\boldsymbol{\psi}(y') \gg \mathbf{0}$ and $\varphi_i(x)|\psi_i(y') - \psi_i(y)| < \varepsilon/2$ for each $i < n$. Define $t'_i = \min\{t_i, \varphi_i(x)\psi_i(y')\}$ for each $i \in \omega$ and note that $|t'_i - t_i| < \varepsilon/2$ for $i < n$. Since $\mathbf{0} \le \mathbf{t}'/\boldsymbol{\psi}(y') \le \boldsymbol{\varphi}(x)$ we can find by the ω-Lelek property an $x' \in U$ such that $\boldsymbol{\varphi}(x') \gg \mathbf{0}$ and

$|\varphi_i(x') - (t'_i/\psi_i(y'))| < \varepsilon/(2\psi_i(y'))$ for each $i < n$. We then have that

$$((x', y'), \boldsymbol{\varphi}(x')\boldsymbol{\psi}(y')) \in G_{\mathbf{0}}^{(\boldsymbol{\varphi}\times\boldsymbol{\psi})\restriction U\times V}$$

and

$$|(\varphi_i(x')\psi_i(y') - t_i| \leq |(\varphi_i(x')\psi_i(y') - t'_i| + |t'_i - t_i| < \epsilon$$

for each $i < n$. In conclusion, we have that $\boldsymbol{\varphi} \times \boldsymbol{\psi}$ is a Lelek function. ∎

Lemma 3.5: *Let X be a complete space and let $\mathcal{T}$ be a topology on X that witnesses the almost zero-dimensionality of X. If Y is a space that contains $(X, \mathcal{T})$ as a dense subspace, then there is an ω-USC function $\boldsymbol{\chi}\colon Y \to \mathbb{I}^\omega$ such that $G_{\mathbf{0}}^{\boldsymbol{\chi}}$ is dense in the graph of $\boldsymbol{\chi}$ and the rule $h(x) = (x, \boldsymbol{\chi}(x))$ defines a homeomorphism h from X to $G_{\mathbf{0}}^{\boldsymbol{\chi}}$.*

Proof. Let Z be the zero-dimensional space $(X, \mathcal{T})$. According to Lemma 2.3 there exists a USC function $\varphi\colon Z \to \mathbb{I}$ such that the rule $f(x) = (x, \varphi(x))$ defines a homeomorphism from X to G_0^φ. We put $\overline{\varphi} = \mathrm{ext}_Y\,\varphi$ and note that this function is USC and that it extends φ. Also observe that G_0^φ, which is the full graph of φ, is dense in the graph of $\overline{\varphi}$. Since X is complete we can find open subsets $O_0, O_1, \ldots$ of $G_0^{\overline{\varphi}}$ such that $G_0^\varphi = \bigcap_{i=0}^\infty O_i$. For each $i \in \omega$ we define $A_i = \{x \in Y : (x, \overline{\varphi}(x)) \notin O_i\}$ and $\psi_i = \mathrm{ext}_Y(\overline{\varphi}\restriction A_i)$. Again, ψ_i is a USC extension of $\overline{\varphi}_i\restriction A_i$. Note that $\bigcup_{i=0}^\infty A_i = Y \setminus Z$. If $x \in Y \setminus A_i$, then $(x, \overline{\varphi}(x))$ is in the open set O_i and hence $\overline{\varphi}(x) > 0$. Since $\overline{\varphi}$ is USC we can find a neighbourhood U of $x \in Y$ and a $t \in (0, \overline{\varphi}(x))$ such that $(U \times (t, \infty)) \cap G_0^{\overline{\varphi}}$ is contained in O_i. Consequently, $\overline{\varphi}(a) \leq t$ for each $a \in U \cap A_i$ and we have that $\psi_i(x) \leq t < \overline{\varphi}(x)$. We have shown that $\{x \in Y : \overline{\varphi}(x) > \psi_i(x)\} = Y \setminus A_i$ for each $i \in \omega$. According to [12, Lemma 4.9] there exists for each $i \in \omega$ a USC function $\chi_i\colon Y \to \mathbb{I}$ such that $Y \setminus A_i = \{x \in Y : \chi_i(x) > 0\}$ and $g_i(x, \overline{\varphi}(x)) = (x, \chi_i(x))$ defines a continuous function from the graph of $\overline{\varphi}$ to the graph of χ_i such that the restriction $g_i\restriction O_i$ is a homeomorphism from O_i to $G_0^{\chi_i}$. Let $\boldsymbol{\chi}\colon Y \to \mathbb{I}^\omega$ be the ω-USC function with components $(\chi_0, \chi_1, \ldots)$ and note that $g(x, \overline{\varphi}(x)) = (x, \boldsymbol{\chi}(x))$ defines a continuous bijection from the graph of $\overline{\varphi}$ to the graph of $\boldsymbol{\chi}$. Note that $Z = \{x \in Y : \boldsymbol{\chi}(x) \gg \mathbf{0}\}$. Since G_0^φ is dense in the graph of $\overline{\varphi}$ we have that $g(G_0^\varphi) = G_{\mathbf{0}}^{\boldsymbol{\chi}}$ is dense in the graph of $\boldsymbol{\chi}$. Since for each i, $g_i\restriction G_0^\varphi$ is a homeomorphism from G_0^φ to $G_0^{\chi_i\restriction Z}$, it is clear that $g\restriction G_0^\varphi$ is a homeomorphism from G_0^φ to $G_{\mathbf{0}}^{\boldsymbol{\chi}}$. Put $h = g \circ f$ and note that the lemma is proved. ∎

LEMMA 3.6: *Let $\boldsymbol{\varphi}\colon X \to \prod_{i\in\omega}[0,s_i]$ be an ω-USC function such that $S = \sum_{i\in\omega} s_i < \infty$. Then $\psi = \sum_{i\in\omega}\varphi_i\colon X \to [0,S]$ is a USC function such that the natural map between the graphs of $\boldsymbol{\varphi}$ and ψ is a homeomorphism. If $\boldsymbol{\varphi}$ is an ω-Lelek function, then ψ is a Lelek function.*

Proof. Let X_1 and X_2 be the set X equipped with the weakest topology that contains the topology of X and that makes $\boldsymbol{\varphi}$ respectively ψ continuous. It is clear that ψ is USC on X and continuous on X_1. To prove that $X_1 = X_2$ it suffices to show that every φ_i is LSC on X_2. Let $x \in X$, $i \in \omega$, and $t < \varphi_i(x)$. Let $k > i$ be such that $\delta = \varphi_i(x) - t - \sum_{j=k}^{\infty} s_j > 0$. By the USC property select a neighbourhood U of x in X such that $\varphi_j(y) < \varphi_j(x) + \delta/k$ for each $y \in U$ and $j \le k$. Note that $V = U \cap \psi^{-1}((\psi(x) - \delta/k, S])$ is a neighbourhood of x in X_2 and let $y \in V$, thus $\psi(x) - \psi(y) - \delta/k < 0$. We then have

$$
\begin{aligned}
\varphi_i(y) &> \varphi_i(y) + \psi(x) - \psi(y) - \delta/k = \varphi_i(y) - \delta/k + \sum_{j=0}^{\infty}(\varphi_j(x) - \varphi_j(y)) \\
&\ge \varphi_i(x) - \delta/k + \sum_{\substack{j=0 \\ j\neq i}}^{k-1}(\varphi_j(x) - \varphi_j(y)) + \sum_{j=k}^{\infty}(\varphi_j(x) - \varphi_j(y)) \\
&\ge \varphi_i(x) - \delta - \sum_{j=k}^{\infty} s_j \\
&= t.
\end{aligned}
\tag{3.1}
$$

Thus we have $\varphi_i(V) \subset (t, s_i]$ and hence every φ_i is LSC on X_2. Since φ_i is USC on X we have that φ_i is continuous on X_2 for $i \in \omega$, thus $X_1 = X_2$, which means that graphs of $\boldsymbol{\varphi}$ and ψ are homeomorphic.

For the second part let $\boldsymbol{\varphi}$ be an ω-Lelek function. In order to show that G_0^{ψ} is dense in L_0^{ψ}, let $x \in X$, U a neighbourhood of x in X, $\varepsilon > 0$, and $t \in [0, \psi(x)]$. Select a $k \in \mathbb{N}$ such that $\delta = \varepsilon - \sum_{i=k}^{\infty} s_i > 0$. Put $r = t/\psi(x)$ if $\psi(x) > 0$ and $r = 1$ if $\psi(x) = 0$, thus $t = r\psi(x)$ and $r \le 1$. There exists a $y \in U$ such that $\boldsymbol{\varphi}(y) \gg \mathbf{0}$ and $|\varphi_i(y) - r\varphi_i(x)| < \delta/k$ for $i < k$. Now we have $\psi(y) > 0$ and

$$
|\psi(y) - t| = |\psi(y) - r\psi(x)| \le \sum_{j=0}^{\infty} |\varphi_j(y) - r\varphi_j(x)| < \sum_{j=0}^{k-1} \delta/k + \sum_{j=k}^{\infty} s_j = \varepsilon. \tag{3.2}
$$

We have shown that ψ is a Lelek function. ∎

4. An extrinsic characterization

Every USC function assumes maxima on compact sets, thus every 1-Lelek function with compact domain is a greatest 1-Lelek function. The proof of the uniqueness of the Lelek fan and of Lelek functions is based on this fact; see [5], [6], and [12, Theorem 6.2]. In this section we will prove the uniqueness of ω-Lelek functions. For $\nu > 1$ it is not true that every ν-Lelek function with compact domain is a greatest ν-Lelek function, but we have the next best thing:

THEOREM 4.1: *If $1 \leq \nu \leq \omega$ and $\boldsymbol{\varphi}$ is a ν-Lelek function with compact domain C, then there is a continuous $\boldsymbol{\beta} : C \to (0,\infty)^\nu$ such that $\boldsymbol{\beta}\cdot\boldsymbol{\varphi}$ is a greatest ν-Lelek function. Moreover, given an $\boldsymbol{\varepsilon} \in (0,\infty)^\nu$ it can be arranged that $M(\log \circ \boldsymbol{\beta}) \leq \boldsymbol{\varepsilon}$.*

Proof. We prove the theorem for $\nu = \omega$, which is the most interesting case. The proof for finite ν is analogous and slightly simpler. Let d be a metric on C such that $d \leq 1$. If $\mathcal{P}$ is a partition of a set X, then a **selection for** $\mathcal{P}$ is a subset S of X such that $|P \cap S| = 1$ for every $P \in \mathcal{P}$. We will construct by recursion clopen partitions $\mathcal{U}_1, \mathcal{U}_2, \ldots$ of C, subsets $S_1 \subset S_2 \subset \cdots$ of C, and continuous functions $\boldsymbol{\beta}^1, \boldsymbol{\beta}^2, \ldots$ from C to $(0,\infty)^\omega$ such that for each $n \in \mathbb{N}$,

(1) if $n > 1$ then $\mathcal{U}_n$ refines $\mathcal{U}_{n-1}$,
(2) $\operatorname{mesh} \mathcal{U}_n \leq 1/n$,
(3) S_n is a selection for $\mathcal{U}_n$,
(4) if $y \in S_n$ then $\boldsymbol{\varphi}(y) \gg \mathbf{0}$,
(5) if $U \in \mathcal{U}_n$ and $y \in U \cap S_n$, then $\xi_n \circ (\boldsymbol{\beta}^n\cdot\boldsymbol{\varphi})\restriction U$ assumes a greatest value at y,
(6) if $n > 1$ then $\log(\boldsymbol{\beta}^n(x)/\boldsymbol{\beta}^{n-1}(x)) \in \prod_{i=0}^\infty[-\varepsilon_i 2^{-n}, 0]$ for each $x \in C$, and
(7) if $n > 1$ then $\boldsymbol{\beta}^n\restriction S_{n-1} = \boldsymbol{\beta}^{n-1}\restriction S_{n-1}$.

We will use the notation $\boldsymbol{\psi}^n = \boldsymbol{\beta}^n\cdot\boldsymbol{\varphi}$ and we note that this function is also ω-Lelek.

For $n = 1$ we put $\mathcal{U}_1 = \{C\}$. Since $G^{\boldsymbol{\varphi}}_{\mathbf{0}}$ is dense in $L^{\boldsymbol{\varphi}}_{\mathbf{0}}$ we may choose an $a \in C$ such that $\boldsymbol{\varphi}(a) \gg \mathbf{0}$ and $\log(M(\varphi_0)/\varphi_0(a)) \leq \varepsilon_0/2$. We put $S_1 = \{a\}$. We can find a continuous map $\alpha : C \to [\varphi_0(a), M(\varphi_0)]$ such that $\alpha(a) = \varphi_0(a)$ and $\varphi_0(x) \leq \alpha(x)$ for every $x \in C$ as follows. Let $C = C_0 \supset C_1 \supset \cdots$ be a clopen neighbourhood basis for a. Since φ_0 is USC we have $\lim_{j\to\infty} M(\varphi_0\restriction C_j) = \varphi_0(a)$, and hence if we let $\alpha(x) = M(\varphi_0 \restriction C_j)$ if $x \in C_j \setminus C_{j+1}$ for $j \in \omega$ and $\alpha(a) = \varphi_0(a)$, then α is a continuous function that meets the requirements. For every $x \in C$ let $\beta_0^1(x) = \varphi_0(a)/\alpha(x)$ and let $\beta_i^1(x) = 1$ when $i > 1$. Thus we have that $\log(\beta_0^1(x)) \in [-\varepsilon_0 2^{-1}, 0]$ for each $x \in C$. We let $\boldsymbol{\beta}^1(x) = (\beta_0^1(x), \beta_1^1(x), \dots)$ for $x \in C$. Observe that the induction hypotheses are satisfied and that for each $x \in C$,

$$\log(\boldsymbol{\beta}^1(x)) \in \prod_{i=0}^{\infty} [-\varepsilon_i 2^{-1}, 0]. \tag{4.1}$$

Assuming that the recursive construction can be performed we verify that the statement of the theorem follows. It is obvious that hypothesis (6) implies that $(\log \circ \beta_i^n)_n$ is a uniform Cauchy sequence for each $i \in \omega$ and hence $\boldsymbol{\beta} = \lim_{n\to\infty} \boldsymbol{\beta}^n \colon C \to (0,\infty)^\omega$ exists and is continuous. Observe also that it follows from formula (4.1) and hypothesis (6) that $\mathbf{1} \geq \boldsymbol{\beta}^1 \geq \boldsymbol{\beta}^2 \geq \cdots \geq \boldsymbol{\beta}$ and $M(\log \circ \boldsymbol{\beta}) \leq \boldsymbol{\varepsilon}$. Note that $\boldsymbol{\beta}\cdot\boldsymbol{\varphi}$ is an ω-Lelek function by Remark 3.1. Let $\delta > 0$ and $m \in \mathbb{N}$. Select an $n \geq m$ such that $1/n < \delta$ and consider the clopen partition $\mathcal{U}_n$. Let $U \in \mathcal{U}_n$ and consider $\{a\} = U \cap S_n$. By hypothesis (7) we have that $\boldsymbol{\beta}^n(a) = \boldsymbol{\beta}^{n+1}(a) = \cdots = \boldsymbol{\beta}(a)$. We have that $\boldsymbol{\beta}^n(x) \geq \boldsymbol{\beta}(x)$ for all $x \in C$ and by hypothesis (5) that $\xi_n \circ (\boldsymbol{\beta}^n\cdot\boldsymbol{\varphi})\restriction U$ assumes a greatest value at a and thus also $\xi_n \circ (\boldsymbol{\beta}\cdot\boldsymbol{\varphi})\restriction U$ assumes a greatest value in a. Since $n \geq m$ also $\xi_m \circ (\boldsymbol{\beta}\cdot\boldsymbol{\varphi})\restriction U$ assumes a greatest value at a. Since $\operatorname{mesh} \mathcal{U}_n < \delta$ we have shown that $\boldsymbol{\beta}\cdot\boldsymbol{\varphi}$ is a greatest ω-Lelek function.

It remains to perform the recursion. Suppose that $\mathcal{U}_n$, $\boldsymbol{\beta}^n$, and S_n have been found for some $n \in \mathbb{N}$. Let $U \in \mathcal{U}_n$ be fixed and let $\mathcal{V}_U$ be a clopen partition of U such that $\operatorname{mesh} \mathcal{V}_U \leq 1/(n+1)$. For every $V \in \mathcal{V}_U$ we will construct a clopen partition $\mathcal{W}_V$ of V and a selection A_V for that partition. We will then define

$$\mathcal{U}_{n+1} = \bigcup \{\mathcal{W}_V : V \in \mathcal{V}_U \text{ and } U \in \mathcal{U}_n\} \tag{4.2}$$

and

$$S_{n+1} = \bigcup \{A_V : V \in \mathcal{V}_U \text{ and } U \in \mathcal{U}_n\}. \tag{4.3}$$

Note that by this construction $\mathcal{U}_{n+1}$ is a clopen partition of C and that hypotheses (1), (2), and (3) are automatically satisfied. Let $V \in \mathcal{V}_U$ be arbitrary.

Case I: $V \cap S_n \neq \emptyset$. Let $\{a\} = V \cap S_n$. We have by hypothesis (4) that $\varphi_n(a) > 0$ and hence $\psi_n^n(a) > 0$. Since $\psi_n^n = \beta_n^n\cdot\varphi_n$ is USC we may assume that V is such that $\log(M(\psi_n^n\restriction V)/\psi_n^n(a)) < \varepsilon_n 2^{-n-1}$. We define $\mathcal{W}_V = \{V\}$ and $A_V = \{a\}$, thereby ensuring that $S_n \subset S_{n+1}$ and that hypothesis (4) is satisfied for $y \in V \cap S_{n+1}$. As above in the base step we can find a continuous $\alpha\colon V \to [\psi_n^n(a), M(\psi_n^n{\restriction}V)]$ such that $\alpha(a) = \psi_n^n(a)$ and $\alpha \geq \psi_n^n$ on V. Define $\boldsymbol{\beta}^{n+1}{\restriction}V$ by $\beta_n^{n+1}{\restriction}V = \psi_n^n(a)(\beta_n^n{\restriction}V)/\alpha$ and $\beta_i^{n+1}{\restriction}V = \beta_i^n{\restriction}V$ for $i \neq n$. Note that $\boldsymbol{\beta}^{n+1}(a) = \boldsymbol{\beta}^n(a)$, which takes care of hypothesis (7). We clearly have that $\psi_n^{n+1}{\restriction}V = (\beta_n^{n+1}{\restriction}V)\cdot(\varphi_n{\restriction}V)$ assumes its maximum at a and that $\log(\beta_n^{n+1}(x)/\beta_n^n(x)) \in [-\varepsilon_n 2^{-n-1}, 0]$ for each $x \in V$. We have for $i < n$ that $\psi_i^{n+1}{\restriction}V = \psi_i^n{\restriction}V$ and by hypothesis (6) that $M(\psi_i^n{\restriction}V) = M(\psi_i^n{\restriction}U) = \psi_i^n(a)$, thus $\xi_{n+1}\circ\boldsymbol{\psi}^{n+1}{\restriction}V$ assumes a greatest value at a, which takes care of hypothesis (5) for V and $n+1$. Clearly, hypothesis (6) is satisfied for $n+1$ and $x \in V$.

Case II: $V \cap S_n = \emptyset$. We construct by recursion finite sequences of positive numbers $\delta_0 > \delta_1 > \cdots > \delta_{n+1}$, clopen partitions $\mathcal{P}_0, \mathcal{P}_1, \ldots, \mathcal{P}_{n+1}$ of V, and subsets $F_0 \subset F_1 \subset \cdots \subset F_{n+1}$ of V such that for each $m \in \{0, 1, \ldots, n+1\}$ we have

(a) F_m is a selection for $\mathcal{P}_m$,
(b) $\boldsymbol{\psi}^n(x) \gg \mathbf{0}$ for each $x \in F_m$,
(c) if $m \geq 1$, $x \in F_{m-1}$, and $0 \leq i \leq n$, then $\psi_i^n(x) \geq \delta_m$, and
(d) if $W \in \mathcal{P}_m$, $y \in W \cap F_m$, and $0 \leq i \leq n$, then $M(\psi_i^n{\restriction}W) < \delta_m$ or $\log(M(\psi_i^n{\restriction}W)/\psi_i^n(y)) < \varepsilon_i 2^{-n-2}$.

For the base step $m = 0$ we choose a δ_0 such that $\delta_0 > M(\varphi_i)$ for each $i \leq n$. Since $\boldsymbol{\psi}^n$ is ω-Lelek we may select an $a \in V$ such that $\boldsymbol{\psi}^n(a) \gg \mathbf{0}$. Put $\mathcal{P}_0 = \{V\}$ and $F_0 = \{a\}$ and note that the induction hypotheses are trivially satisfied.

Now assume that δ_m, $\mathcal{P}_m$, and F_m have been found for some $m \in \{0, \ldots, n\}$. Since V is compact both $\mathcal{P}_m$ and F_m are finite sets. Thus we can find with induction hypothesis (b) a $\delta_{m+1} \in (0, \delta_m)$ such that $\delta_{m+1} \leq \psi_i^n(x)$ for every $x \in F_m$ and $i \leq n$, thereby satisfying (c) for $m+1$. Select for every $x \in V$ a clopen neighbourhood B_x in V such that for each $i \leq n$,

$$\text{if} \quad \psi_i^n(x) = 0 \quad \text{then} \quad M(\psi_i^n{\restriction}B_x) < \delta_{m+1} \tag{4.4}$$

and

$$\text{if} \quad \psi_i^n(x) > 0 \quad \text{then} \quad \log(M(\psi_i^n{\restriction}B_x)/\psi_i^n(x)) < \varepsilon_i 2^{-n-2}. \tag{4.5}$$

Since V is compact we can find a finite subset N of V with $V = \bigcup\{B_x : x \in N\}$. We may assume that $F_m \subset N$. Now shrink the cover $\{B_x : x \in N\}$ to a clopen partition $\mathcal{P}_{m+1} = \{P_x : x \in N\}$ of V such that $x \in P_x \subset B_x$ for each $x \in N$. Observe that it follows from (4.4) and (4.5) that hypothesis (d) is satisfied whenever $W \in \mathcal{P}_{m+1}$ and $y \in W \cap N$. Since $\boldsymbol{\psi}^n$ is ω-Lelek we can approximate every $x \in N$ with an $x' \in P_x$ such that $\boldsymbol{\psi}^n(x') \gg \mathbf{0}$ and $\log(M(\psi_i^n \restriction W)/\psi_i^n(x')) < \varepsilon_i 2^{-n-2}$ whenever $\log(M(\psi_i^n \restriction W)/\psi_i^n(x)) < \varepsilon_i 2^{-n-2}$ and $i \leq n$. We put $F_{m+1} = \{x' : x \in N\}$ and note that (a), (b), and (d) are satisfied for $m+1$. Of course we choose $x' = x$ if it happens that already $\boldsymbol{\psi}^n(x) \gg \mathbf{0}$ so that hypothesis (b) implies that $F_m \subset F_{m+1}$. This completes the construction of the δ, $\mathcal{P}$, and F sequences.

Proceeding with Case II, we define a function $f\colon F_{n+1} \to F_{n+1}$ as follows. Let $y \in F_{n+1}$ and put $T = \{\psi_i^n(y) : 0 \leq i \leq n\}$. If $T \cap [0, \delta_{n+1}) = \emptyset$ then we define $f(y) = y$. Now let $T \cap [0, \delta_{n+1}) \neq \emptyset$ and hence $T \cap [\delta_{n+1}, \delta_0)$ has at most n elements. Since $\{[\delta_{k+1}, \delta_k) : 0 \leq k \leq n\}$ is a partition of $[\delta_{n+1}, \delta_0)$ into $n+1$ subintervals, we can select a $k \in \{0, 1, \ldots, n\}$ such that

$$T \cap [\delta_{k+1}, \delta_k) = \emptyset. \tag{4.6}$$

Let P be the element of $\mathcal{P}_k$ that contains y and let z be the unique element of $P \cap F_k$. Since $F_k \subset F_{n+1}$ we may define $f(y) = z$. By hypothesis (c) for $m = k+1$ we have $\psi_i^n(z) \geq \delta_{k+1} \geq \delta_{n+1}$ for each $i \leq n$ and hence $f(f(y)) = f(z) = z = f(y)$. Let us consider an arbitrary $i \leq n$. If $\psi_i^n(y) < \delta_{k+1}$ then we have $\psi_i^n(y) < \psi_i^n(z)$. Now let $\psi_i^n(y) \geq \delta_{k+1}$. Then by formula (4.6) we have $M(\psi_i^n \restriction P) \geq \psi_i^n(y) \geq \delta_k$ and hence, according to hypothesis (d) for $m = k$,

$$\log(\psi_i^n(y)/\psi_i^n(z)) \leq \log(M(\psi_i^n \restriction P)/\psi_i^n(z)) < \varepsilon_i 2^{-n-2}. \tag{4.7}$$

We can summarize as follows:

$$f \circ f = f, \tag{4.8}$$

$$\psi_i^n(z) \geq \delta_{n+1} \quad \text{for } i \leq n \text{ and } z \in f(F_{n+1}), \text{ and} \tag{4.9}$$

$$\log(\psi_i^n(y)/\psi_i^n(f(y))) < \varepsilon_i 2^{-n-2} \quad \text{for } i \leq n \text{ and } y \in F_{n+1}. \tag{4.10}$$

We define $A_V = f(F_{n+1})$ and

$$\mathcal{W}_V = \left\{ \bigcup \{P \in \mathcal{P}_{n+1} : P \cap f^{-1}(z) \neq \emptyset\} : z \in A_V \right\}. \tag{4.11}$$

Note that $\mathcal{W}_V$ is a clopen partition of V and that A_V is a corresponding selection because f is a retraction. Recall that $\mathcal{U}_{n+1}$ and S_{n+1} are given by (4.2) and (4.3). Observe that (b) implies that hypothesis (4) is satisfied for $n+1$.

Let $W \in \mathcal{W}_V$ be arbitrary and let $\{z\} = W \cap A_V$. We will define $\boldsymbol{\beta}^{n+1} \restriction W$. Consider an $x \in W$. Then there exist a $P \in \mathcal{P}_{n+1}$ and a $y \in P \cap F_{n+1}$ such that

$x \in P$ and $f(y) = z$. Let $i \leq n$. If $\psi_i^n(x) < \delta_{n+1}$, then by (4.9) $\psi_i^n(x) < \psi_i^n(z)$. If $\psi_i^n(x) \geq \delta_{n+1}$, then by hypothesis (d) for $m = n+1$ we have

$$\log(\psi_i^n(x)/\psi_i^n(y)) \leq \log(M(\psi_i^n{\restriction}P)/\psi_i^n(y)) < \varepsilon_i 2^{-n-2}, \tag{4.12}$$

which combines with (4.10) to

$$\log(\psi_i^n(x)/\psi_i^n(z)) < \varepsilon_i 2^{-n-1}. \tag{4.13}$$

Thus (4.13) is valid for every $i \leq n$ and we have

$$\log(M(\psi_i^n{\restriction}W)/\psi_i^n(z)) \leq \varepsilon_i 2^{-n-1}. \tag{4.14}$$

As above, we find for every $i \leq n$ a continuous $\alpha_i \colon W \to [\psi_i^n(z), M(\psi_i^n{\restriction}W)]$ such that $\alpha_i(z) = \psi_i^n(z)$ and $\alpha_i \geq \psi_i^n$ on W. Define $\boldsymbol{\beta}^{n+1}{\restriction}W$ by $\beta_i^{n+1}{\restriction}W = \psi_i^n(z)(\beta_i^n{\restriction}W)/\alpha_i$ for $i \leq n$ and $\beta_i^{n+1}{\restriction}W = \beta_i^n{\restriction}W$ for $i > n$. We clearly have that $M(\psi_i^{n+1}{\restriction}W) = M((\beta_i^{n+1}{\restriction}W)\cdot(\varphi_i{\restriction}W)) = \psi_i^n(z) = \psi_i^{n+1}(z)$ for $i \leq n$, which takes care of hypothesis (5) for $n+1$. It follows from (4.14) that $\log(\beta_i^{n+1}(x)/\beta_i^n(x)) \in [-\varepsilon_i 2^{-n-1}, 0]$ for each $x \in W$ and $i \leq n$. Since $\beta_i^{n+1}{\restriction}W = \beta_i^n{\restriction}W$ for $i > n$ we have hypothesis (6). Hypothesis (7) applies only to Case I. The proof is complete. ∎

THEOREM 4.2: *If $1 \leq \nu \leq \omega$ then any two ν-Lelek functions with compact domains are homeomorphic.*

Proof. Again we give the proof for $\nu = \omega$. Let $\boldsymbol{\varphi} \colon C \to [0,\infty)^\omega$ and $\boldsymbol{\psi} \colon D \to [0,\infty)^\omega$ be greatest ω-Lelek functions with C and D compact, using Theorem 4.1. Select metrics on C and D that are bounded by 1. We construct by induction sequences of clopen partitions $\mathcal{U}_0, \mathcal{U}_1, \ldots$ of C and homeomorphisms $h_0, h_1, \ldots$ from C to D such that, for each $n \in \omega$,

(1) if $n \geq 1$ then $\mathcal{U}_n$ refines $\mathcal{U}_{n-1}$,
(2) $\operatorname{mesh} \mathcal{U}_n \leq 2^{-n}$,
(3) $\operatorname{mesh} h_n[\mathcal{U}_n] \leq 2^{-n}$,
(4) if $n \geq 1$ then $h_n(U) = h_{n-1}(U)$ for each $U \in \mathcal{U}_{n-1}$,
(5) for every $U \in \mathcal{U}_n$, $\xi_{n+1} \circ \boldsymbol{\varphi}$ and $\xi_{n+1} \circ \boldsymbol{\psi}$ assume greatest values on U respectively $h_n(U)$, and
(6) if $n \geq 1$, $U \in \mathcal{U}_{n-1}$, and $V \in \mathcal{U}_n$ such that $V \subset U$, then for each $i < n$, $|\log(\gamma_i(V)/\gamma_i(U))| < 2^{-n}$, where

$$\gamma_i(U) = \frac{M(\psi_i \circ h_{n-1}{\restriction}U)}{M(\varphi_i{\restriction}U)} \quad \text{and} \quad \gamma_i(V) = \frac{M(\psi_i \circ h_n{\restriction}V)}{M(\varphi_i{\restriction}V)}. \tag{4.15}$$

The domain of a Lelek function cannot have isolated points, thus C and D are Cantor sets. Let $h_0 : C \to D$ be some homeomorphism and put $\mathcal{U}_0 = \{C\}$, and note that the induction hypotheses for $n = 0$ are satisfied. Assume now that h_n and $\mathcal{U}_n$ have been constructed. Let $U \in \mathcal{U}_n$. We will construct a clopen partition $\mathcal{V}_U$ of U and a homeomorphism $f_U : U \to h_n(U)$ which will produce $\mathcal{U}_{n+1}$ and h_{n+1} as follows:

$$\mathcal{U}_{n+1} = \bigcup_{U \in \mathcal{U}_n} \mathcal{V}_U \quad \text{and} \quad h_{n+1} = \bigcup_{U \in \mathcal{U}_n} f_U. \tag{4.16}$$

Note that by this construction hypotheses (1) and (4) are automatically satisfied.

By hypothesis (5) we have that $\xi_{n+1}(\boldsymbol{\varphi}(a)) = M(\xi_{n+1} \circ \boldsymbol{\varphi} \restriction U)$ and $\xi_{n+1}(\boldsymbol{\psi}(b)) = M(\xi_{n+1} \circ \boldsymbol{\psi} \restriction h_n(U))$ for some $a \in U$ and $b \in h_n(U)$. By the greatest ω-Lelek property select clopen partitions $\mathcal{A} = \{A_0, \ldots, A_k\}$ and $\mathcal{B} = \{B_0, \ldots, B_l\}$ of U respectively $h_n(U)$ such that $a \in A_0$, $b \in B_0$, $\operatorname{mesh} \mathcal{A} \le 2^{-n-1}$, $\operatorname{mesh} \mathcal{B} \le 2^{-n-1}$, $\xi_{n+2} \circ \boldsymbol{\varphi}$ assumes a greatest value on every A_m, and $\xi_{n+2} \circ \boldsymbol{\psi}$ assumes a greatest value on every B_j. Let $a' \in A_0$ and $b' \in B_0$ be such that $\xi_{n+2}(\boldsymbol{\varphi}(a')) = M(\xi_{n+1} \circ \boldsymbol{\varphi} \restriction A_0)$ and $\xi_{n+2}(\boldsymbol{\psi}(b')) = M(\xi_{n+1} \circ \boldsymbol{\psi} \restriction B_0)$. Then we have $\xi_{n+1}(\boldsymbol{\varphi}(a')) = \xi_{n+1}(\boldsymbol{\varphi}(a))$ and $\xi_{n+1}(\boldsymbol{\psi}(b')) = \xi_{n+1}(\boldsymbol{\psi}(b))$ are the greatest values of $\xi_{n+1} \circ \boldsymbol{\varphi} \restriction U$ respectively $\xi_{n+1} \circ \boldsymbol{\psi} \restriction h_n(U)$.

Let $m \in \{1, \ldots, k\}$. If $i \in \{0, \ldots, n\}$ then we have

$$0 < \gamma_i(U) M(\varphi_i \restriction A_m) \le \gamma_i(U) M(\varphi_i \restriction U) = M(\psi_i \circ h_n \restriction U) = \psi_i(b'). \tag{4.17}$$

The ω-Lelek property allows us to select distinct points $b_m \in B_0 \setminus \{b'\}$, close to b', such that for each $i \le n$,

$$\left| \log \frac{\psi_i(b_m)}{\gamma_i(U) M(\varphi_i \restriction A_m)} \right| < 2^{-n-2}. \tag{4.18}$$

Choose disjoint clopen sets $V_1, \ldots, V_k$ in D such that for each m, $b_m \in V_m \subset B_0 \setminus \{b'\}$,

$$\log(M(\psi_i \restriction V_m)/\psi_i(b_m)) < 2^{-n-2} \tag{4.19}$$

for each $i \le n$ by upper semi-continuity, and $\xi_{n+2} \circ \boldsymbol{\psi}$ assumes a greatest value on V_m by the greatest Lelek property. We will have that $\{A_1, \ldots, A_k\} \subset \mathcal{V}_U$ and $h_{n+1}(A_m) = f_U(A_m) = V_m$, and hence $\gamma_i(A_m)$ will have the value

$M(\psi_i{\restriction}V_m)/M(\varphi_i{\restriction}A_m)$. Note that for each $i \leq n$,

$$\begin{aligned}\left|\log\frac{\gamma_i(A_m)}{\gamma_i(U)}\right| &= \left|\log\frac{M(\psi_i{\restriction}V_m)}{\gamma_i(U)M(\varphi_i{\restriction}A_m)}\right| \\ &\leq \left|\log\frac{M(\psi_i{\restriction}V_m)}{\psi_i(b_m)}\right| + \left|\log\frac{\psi_i(b_m)}{\gamma_i(U)M(\varphi_i{\restriction}A_m)}\right| \\ &< 2^{-n-1},\end{aligned} \tag{4.20}$$

satisfying hypothesis (6) for $A_m \in \mathcal{U}_{n+1}$ and $m \in \{1, \ldots, k\}$. Note that hypothesis (5) is also satisfied for such an A_m.

Conversely, we can find disjoint clopen sets $W_1, \ldots, W_l$ contained in $A_0 \setminus \{a'\}$ such that, for each $m \in \{1, \ldots, l\}$, $\xi_{n+2} \circ \boldsymbol{\varphi}$ assumes a greatest value on W_m and

$$\left|\log\frac{M(\psi_i{\restriction}B_m)}{\gamma_i(U)M(\varphi_i{\restriction}W_m)}\right| < 2^{-n-1} \tag{4.21}$$

for each $i \leq n$. Put $A_0' = A_0 \setminus \bigcup_{i=1}^{l} W_i$ and $B_0' = B_0 \setminus \bigcup_{j=1}^{k} V_j$. We define $\mathcal{V}_U = \{A_0', A_1, \ldots, A_k, W_1, \ldots, W_l\}$ and we let $h_{n+1}{\restriction}U = f_U \colon U \to h_n(U)$ be a homeomorphism with $f_U(A_m) = V_m$ and $f_U(W_j) = B_j$ for $1 \leq m \leq k$, $1 \leq j \leq l$ and $f_U(A_0') = B_0'$. Since $\mathcal{V}_U$ and $f_U[\mathcal{V}_U]$ refine $\mathcal{A}$ respectively $\mathcal{B}$ we have that hypotheses (2) and (3) are satisfied for $n+1$. It follows from (4.21) that hypothesis (6) is satisfied for every $W_j \in \mathcal{U}_{n+1}$. Also, hypothesis (5) is satisfied for each W_j. Note that

$$\gamma_i(A_0') = \frac{M(\psi_i{\restriction}B_0')}{M(\varphi_i{\restriction}A_0')} = \frac{\psi_i(b')}{\varphi_i(a')} = \gamma_i(U) \tag{4.22}$$

for each $i \leq n$. Thus hypothesis (6) is verified for A_0'. Also, hypothesis (5) is clearly satisfied and the induction is complete.

Obviously, $h = \lim_{n\to\infty} h_n$ is a homeomorphism $C \to D$. Let $i \in \omega$ be arbitrary. Define for each $n > i$ the continuous function $\alpha_i^n \colon C \to (0, \infty)$ by

$$\alpha_i^n(x) = \gamma_i(U) \quad \text{for } x \in U \in \mathcal{U}_n. \tag{4.23}$$

Note that by hypothesis (6),

$$|\log(\alpha_i^n(x)/\alpha_i^{n-1}(x))| < 2^{-n} \tag{4.24}$$

for each $x \in C$ and $n > i+1$. Thus $(\log \circ \alpha_i^n)_{n>i}$ is a uniform Cauchy sequence of continuous functions into $\mathbb{R}$ and hence $\alpha_i = \lim_{n\to\infty} \alpha_i^n \colon C \to (0, \infty)$ exists and is continuous. Now, let $x \in C$ and select for each $n > i$ a $U_n \in \mathcal{U}_n$ with $x \in U_n$. Since $h_n(U_n) = h_k(U_n)$ for all $k > n$ we have $h(x) \in h_n(U_n)$.

By upper semi-continuity and $\operatorname{diam} U_n \leq 2^{-n}$, $\operatorname{diam} h_n(U_n) \leq 2^{-n}$ we have $\lim_{n\to\infty} M(\varphi_i{\restriction}U_n) = \varphi_i(x)$ and $\lim_{n\to\infty} M(\psi_i \circ h_n{\restriction}U_n) = \psi_i(h(x))$. Thus by (4.23) and (4.15) we have, for each $x \in C$,

$$\begin{aligned}\alpha_i(x)\varphi_i(x) &= \lim_{n\to\infty} \alpha_i^n(x) M(\varphi_i{\restriction}U_n) = \lim_{n\to\infty} \gamma_i(U_n) M(\varphi_i{\restriction}U_n)\\ &= \lim_{n\to\infty} M(\psi_i \circ h_n{\restriction}U_n) = \psi_i(h(x)).\end{aligned} \tag{4.25}$$

In conclusion, $\boldsymbol{\alpha}\cdot\boldsymbol{\varphi} = \boldsymbol{\psi} \circ h$. ∎

If we combine Theorem 4.2 with Remark 2.8, then we have a characterization theorem for $\mathfrak{E}_{\mathrm{c}}^{\omega}$:

Theorem 4.3: *A space X is homeomorphic to $\mathfrak{E}_{\mathrm{c}}^{\omega}$ if and only if there is an ω-Lelek function $\boldsymbol{\varphi}$ with compact domain such that $X \approx G_{\mathbf{0}}^{\boldsymbol{\varphi}}$.*

5. Intrinsic characterizations

We will “internalize” Theorem 4.3, that is, we will convert that theorem into theorems that refer only to topological properties that are internal to the space considered. We first need to add a feature to Lemma 2.3.

Lemma 5.1: *Let X be a space, let $\mathcal{U}$ be an open covering of X, and let Z be zero-dimensional space that contains X as a subset (but not necessarily as a subspace). If Z is a witness to the almost zero-dimensionality of X, then there exists a USC function $\varphi\colon Z \to \mathbb{I}$ such that the map $h\colon X \to G_0^{\varphi}$ that is defined by the rule $h(x) = (x, \varphi(x))$ is a homeomorphism and for every $t \in (0,1]$ the set $\varphi^{-1}([t,1])$ can be covered using only finitely many elements of $\mathcal{U}$.*

Proof. According to Lemma 2.3 there is a USC function $\psi\colon Z \to \mathbb{I}$ such that the map $f\colon X \to G_0^{\psi}$ that is defined by the rule $f(x) = (x, \psi(x))$ is a homeomorphism. Without loss of generality we may assume that $\mathcal{U}$ has the form

$$\{O_n \cap \psi^{-1}((t_n, 1]) : n \in \omega\}, \tag{5.1}$$

where O_n is clopen in Z and $t_n \in (0,1)$ for $n \in \omega$. We define for $n \in \omega$ the function $\chi_n\colon Z \to [0, 2^{-n-1}]$ by

$$\chi_n(x) = \begin{cases} 0, & \text{if } x \in Z \setminus O_n;\\ 2^{-n-1}\max\{0, \psi(x) - t_n\}, & \text{if } x \in O_n.\end{cases} \tag{5.2}$$

Put $\varphi = \sum_{n=1}^{\infty} \chi_n \colon Z \to \mathbb{I}$. Note that $\chi_n \leq 2^{-n-1}\psi$ for each $n \in \mathbb{N}$, thus $\varphi \leq \psi$ and $\{x \in Z : \varphi(x) > 0\} \subset \{x \in Z : \psi(x) > 0\} = X$. If $x \in X$ then $x \in U_n$ for some n, thus $\chi_n(x) > 0$ and $\varphi(x) > 0$. We have that $X = \{x \in Z : \varphi(x) > 0\}$ and hence h is a bijection from X to G_0^{φ}. It is clear that χ_n is USC on Z and continuous on X because ψ has those properties and O_n is clopen in Z. If $x, y \in U_n = O_n \cap \psi^{-1}((t_n, 1])$ then $\chi_n(x) - \chi_n(y) = 2^{-n-1}(\varphi(x) - \varphi(y))$, so the natural map from U_n to the graph of $\chi_n \colon Z \to [0, 2^{-n-1}]$ is an imbedding. Since $\mathcal{U}$ is an open cover we have that the natural map from X to the graph of $\boldsymbol{\chi} = (\chi_0, \chi_1, \dots)$ is also an imbedding. According to Lemma 3.6, $\varphi \colon Z \to \mathbb{I}$ is USC and the bijection $h \colon X \to G_0^{\varphi}$ is a homeomorphism. Note that the definition of χ_n is such that, whenever $x \notin U_n$, then $\chi_n(x) = 0$. Thus if $x \notin \bigcup_{i=0}^{n-1} U_i$ then $\varphi(x) \leq \sum_{i=n}^{\infty} 2^{-i-1} = 2^{-n}$. We have that $\varphi^{-1}([t, 1])$ can be covered using only finitely many elements of $\mathcal{U}$ for each positive t. ∎

If $\mathcal{T}_0, \mathcal{T}_1, \dots$ is a sequence of topologies on a set X, then $\bigvee_{i \in \omega} \mathcal{T}_i$ denotes the topology on X that is generated by the subbasis $\bigcup_{i \in \omega} \mathcal{T}_i$.

Remark 5.2: If every $X_i = (X, \mathcal{T}_i)$ is separable metric, then so is X with the topology $\bigvee_{i \in \omega} \mathcal{T}_i$ because that space is homeomorphic to the diagonal in $\prod_{i \in \omega} X_i$. Morover, if there is a topology $\mathcal{W}$ on X that witnesses the almost zero-dimensionality of every $\mathcal{T}_i$, then it is easily verified that $\mathcal{W}$ is also a witness to $\bigvee_{i \in \omega} \mathcal{T}_i$.

THEOREM 5.3: *A nonempty space X is homeomorphic to $\mathfrak{E}_c^{\omega}$ if and only if there exist topologies $\mathcal{W}, \mathcal{T}_0, \mathcal{T}_1, \dots$ on X and covers $\mathcal{U}_i \subset \mathcal{T}_i$ of X for $i \in \omega$ such that*

1. *the given topology $\mathcal{S}$ on X is equal to $\bigvee_{i \in \omega} \mathcal{T}_i$;*
2. *$Z = (X, \mathcal{W})$ is a witness to the almost zero-dimensionality of every $X_i = (X, \mathcal{T}_i)$;*
3. *for every $i \in \omega$ the space X_i is $\left(\bigvee_{j \neq i} \mathcal{T}_j\right)$-cohesive; and*
4. *if $U_i \in \mathcal{U}_i$ is such that $\{U_i : i \in \omega\}$ has the finite intersection property, then the sequence $(\bigcap_{j=0}^{i} U_j)_{i \in \omega}$ converges to a point in Z.*

Proof. Represent $\mathfrak{E}_c^{\omega}$ by the standard model $G_{\mathbf{0}}^{\boldsymbol{\psi}}$ as in Remark 2.8: C is a Cantor set, $\varphi \colon C \to [0, \infty)$ is a Lelek function, and $\boldsymbol{\psi} \colon C^{\omega} \to [0, \infty)^{\omega}$ is given by the rule $\psi_i(x_0, x_1, \dots) = \varphi(x_i)$. Define the projections $\pi \colon G_{\mathbf{0}}^{\boldsymbol{\psi}} \to C^{\omega}$, $\pi_i \colon G_{\mathbf{0}}^{\boldsymbol{\psi}} \to C$, and $p_i \colon G_{\mathbf{0}}^{\boldsymbol{\psi}} \to G_0^{\varphi}$ by $\pi(x, \boldsymbol{\psi}(x)) = x$, $\pi_i(x, \boldsymbol{\psi}(x)) = x_i$, and $p_i(x, \boldsymbol{\psi}(x)) = (x_i, \varphi(x_i))$ for $i \in \omega$. We let $\mathcal{W}$ be the zero-dimensional topology on $G_0^{\boldsymbol{\psi}}$ that is generated by π and we let $\mathcal{T}_i$ be generated by π and p_i for $i \in \omega$. It is clear that the conditions (1) and (2) are now satisfied. To verify condition (3) let $(x, \boldsymbol{\psi}(x)) \in G_{\mathbf{0}}^{\boldsymbol{\psi}}$ and let $i \in \omega$. Since $\mathfrak{E}_{\mathrm{c}} \approx G_0^{\varphi}$ is cohesive there is an open neighbourhood U of x_i in that space such that no nonempty clopen subset of G_0^{φ} is contained in U. Consider the $\mathcal{T}_i$-open neighbourhood $p_i^{-1}(U)$ of x in $G_{\mathbf{0}}^{\boldsymbol{\varphi}}$. Let $O \in \bigvee_{j \neq i} \mathcal{T}_j$ and let P be a nonempty clopen subset of $(O, \mathcal{T}_i)$. Select a $(y, \boldsymbol{\psi}(y)) \in P$ and define the set

$$Y = \{(z, \boldsymbol{\psi}(z)) : z_j = y_j \text{ for every } j \neq i\} \subset G_{\mathbf{0}}^{\boldsymbol{\psi}} \tag{5.3}$$

and note that $p_j \restriction Y$ is constant for $j \neq i$, thus on Y the topology $\bigvee_{j \neq i} \mathcal{T}_j$ coincides with $\mathcal{W}$. Thus we may select a clopen subset K of C such that $y \in Y \cap \pi_i^{-1}(K) \subset O$. Consequently, we have that $P' = P \cap Y \cap \pi_i^{-1}(K)$ is a clopen subset of $(Y, \mathcal{T}_i)$ that contains y. Note that $p_i \restriction Y$ is a homeomorphism from $(Y, \mathcal{T}_i)$ to G_0^{φ} and hence $p_i(P')$ is a nonempty clopen subset of G_0^{φ}. By the choice of U we have that $p_i(P') \not\subset U$ and hence $P \not\subset \pi_i^{-1}(U)$. We have verified condition (3). Turning to condition (4) we choose a compatible metric d on C^{ω}. For $i \in \omega$ we define

$$\begin{aligned} \mathcal{U}_i = \{\pi^{-1}(W \cap \psi_i^{-1}((\varepsilon, \infty))) :\ & \text{for some } \varepsilon > 0 \text{ and } W \in \mathcal{W} \\ & \text{with } \operatorname{diam} W < 2^{-i}\}. \end{aligned} \tag{5.4}$$

Note that $\mathcal{U}_i$ covers $G_{\mathbf{0}}^{\boldsymbol{\psi}}$ and that $\mathcal{U}_i \subset \mathcal{T}_i$. Let $U_i \in \mathcal{U}_i$ for $i \in \omega$ such that $\{U_i : i \in \omega\}$ has the finite intersection property. Since C^{ω} is compact and $\operatorname{diam} \pi(U_i) < 2^{-i}$, we have that $(\bigcap_{j=0}^{i} \pi(U_j))_{i \in \omega}$ converges to a point z in C^{ω}. Since for each $i \in \omega$ there is an $\varepsilon_i > 0$ such that $\pi(U_i) \subset \psi_i^{-1}((\varepsilon_i, \infty))$ and since $\psi_i^{-1}([\varepsilon_i, \infty))$ is closed in C^{ω} by upper semi-continuity, we have that $\psi_i(z) \geq \varepsilon_i$. We may conclude that $\boldsymbol{\psi}(z) \gg \mathbf{0}$, thus $(z, \boldsymbol{\psi}(z)) \in G_{\mathbf{0}}^{\boldsymbol{\psi}}$ and condition (4) is verified.

Now assume that X is an arbitrary space with the listed properties. Let $i \in \omega$. Using Lemma 5.1 and assumption (2) we can select a USC function $\chi_i \colon Z \to (0, 1]$ such that the natural map from X_i to $G_0^{\chi_i}$ is a homeomorphism and for every $t \in (0, 1]$ the set $\chi_i^{-1}([t, 1])$ can be covered using only finitely many elements of $\mathcal{U}_i$. With assumption (3) and Lemma 2.10 we can find a USC function $\varphi_i \colon Z \to (0, 1]$ such that the natural map from X_i to $G_0^{\varphi_i}$ is a homeomorphism, $\varphi_i \leq \chi_i$, and for every $U \in \bigvee_{j \neq i} \mathcal{T}_j \setminus \{\emptyset\}$, we have that $\varphi_i \restriction U$ is a Lelek function. Put $\boldsymbol{\varphi} = (\varphi_0, \varphi_1, \dots)$ and note that the natural map from X to $G_{\mathbf{0}}^{\boldsymbol{\varphi}}$, when seen as a subspace of $Z \times \mathbb{I}^\omega$, is a homeomorphism by assumption (1).

Claim 1: *$\boldsymbol{\varphi}$ is an ω-Lelek function.*

Proof. We prove by induction that for $n \in \mathbb{N}$,

$$\xi_n \circ \boldsymbol{\varphi}\restriction U \text{ is } n\text{-Lelek for every } U \in \textstyle\bigvee_{i\geq n} \mathcal{T}_i \setminus \{\emptyset\}. \tag{5.5}$$

Since the graph of $\boldsymbol{\varphi}$ equals $G_{\mathbf{0}}^{\boldsymbol{\varphi}}$ the claim follows.

The basis step $n = 1$ is trivially valid because $\xi_1 \circ \boldsymbol{\varphi} = \varphi_0$. Assume that the induction hypothesis (5.5) is satisfied for some $n \in \mathbb{N}$. Let $U \in \bigvee_{i>n} \mathcal{T}_i$, $x \in U$, and $\mathbf{t} \in \mathbb{I}^{n+1}$ be such that $\mathbf{t} \leq \xi_{n+1}(\boldsymbol{\varphi}(x))$. Let O be an open neighbourhood of x in Z and let $\varepsilon > 0$ be arbitrary. Define

$$V = U \cap \bigcap_{i=0}^{n-1} \varphi_i^{-1}((\varphi_i(x) - \varepsilon/2, 1]) \tag{5.6}$$

and note that $x \in V$ and $V \in \bigvee_{i\neq n} \mathcal{T}_i$. Since $\varphi_n\restriction V$ is a Lelek function we can find a $y \in V \cap O$ such that $|\varphi_n(y) - t_n| < \varepsilon$. Define $W = U \cap \varphi_n^{-1}((t_n - \varepsilon, t_n + \varepsilon))$ and note that $y \in W \cap O$ and $W \in \bigvee_{i\geq n} \mathcal{T}_i$. Put $t'_i = \min\{t_i, \varphi_i(y)\}$ for $i < n$ and thus $\mathbf{t}' = (t'_0, \dots, t'_{n-1}) \leq \xi_n(\boldsymbol{\varphi}(y))$. By the induction hypothesis $\xi_n \circ \boldsymbol{\varphi}\restriction W$ is an n-Lelek function, thus there is a $z \in W \cap O$ such that $|\varphi_i(z) - t'_i| < \varepsilon/2$ for $i < n$. Let $i < n$. Since $y \in V$ we have $\varphi_i(y) > \varphi_i(x) - \varepsilon/2 \geq t_i - \varepsilon/2$, thus $|t'_i - t_i| < \varepsilon/2$. We now have that $|\varphi_i(z) - t_i| \leq |\varphi_i(z) - t'_i| + |t'_i - t_i| < \varepsilon$. Since $z \in W \cap O$ we also have $|\varphi_n(z) - t_n| < \varepsilon$ and $z \in U \cap O$. Thus $(z, \xi_{n+1}(\boldsymbol{\varphi}(z)))$ approximates $(x, \mathbf{t})$ and we may conclude that $\xi_{n+1} \circ \boldsymbol{\varphi}\restriction U$ is an $(n+1)$-Lelek function. ∎

Let $\mathcal{A}$ consist of all finite intersections of sets of the form $A_i(q) = \{x \in Z : \varphi_i(z) \geq q\}$ for $i \in \omega$ and $q \in \mathbb{Q}$. Note that $\mathcal{A}$ is a countable collection of closed subsets of Z. By a standard Stone space construction we can extend Z to a zero-dimensional compactification C such that, for all $F_1, F_2 \in \mathcal{A}$, we have $\overline{F_1 \cap F_2} = \overline{F}_1 \cap \overline{F}_2$. We define the ω-USC function $\boldsymbol{\psi}\colon C \to \mathbb{I}^\omega$ by $\psi_i(x) = (\operatorname{ext}_C \varphi_i)(x)$ for $i \in \omega$ and $x \in C$. Note that if $\psi_i(x) > q$, then by the definition of ext we have that x is in the closure of $A_i(q)$. This leads immediately to:

Claim 2: *If $x \in C$ and $\mathbf{q} \in \mathbb{Q}^\omega$ are such that $\mathbf{q} \ll \boldsymbol{\psi}(x)$, then x is in the closure in C of $\bigcap_{i=0}^n A_i(q_i)$ for each $n \in \omega$.*

We have $\psi_i \restriction Z = \varphi_i > 0$, so the natural map from X_i to the graph of $\psi_i \restriction Z$ is a homeomorphism and the graph of $\boldsymbol{\psi} \restriction Z = \boldsymbol{\varphi}$ is contained in $G_{\mathbf{0}}^{\boldsymbol{\psi}}$. Then $\boldsymbol{\psi}$ is an ω-Lelek function by the following result.

Claim 3: *The graph of $\boldsymbol{\varphi}$ is dense in $L_{\mathbf{0}}^{\boldsymbol{\psi}}$.*

Proof. Let $x \in C$, $n \in \mathbb{N}$, $\varepsilon > 0$, U an open neighbourhood of x in C, and $\mathbf{t} \in \mathbb{I}^n$ such that $\mathbf{t} \leq \xi_n(\boldsymbol{\psi}(x))$. Select for each $i < n$ a $q_i \in \mathbb{Q} \cap (\psi_i(x) - \varepsilon/2, \psi_i(x))$. According to Claim 2 there is a $y \in U \cap \bigcap_{i=0}^{n-1} A_i(q_i)$. Put, for $i < n$, $t'_i = \min\{t_i, \varphi_i(y)\}$ and note that $|t'_i - t_i| < \varepsilon/2$ because $\varphi_i(y) \geq q_i > \psi_i(x) - \varepsilon/2 \geq t_i - \varepsilon/2$. Since $\mathbf{t}' \leq \xi_n(\boldsymbol{\varphi}(y))$ and $\boldsymbol{\varphi}$ is an ω-Lelek function by Claim 1, we have that there is a $z \in U$ such that $|\varphi_i(z) - t'_i| < \varepsilon/2$ for each $i < n$. Thus $|\varphi_i(z) - t_i| < \varepsilon$ for each $i < n$ and we have shown that the graph of $\boldsymbol{\varphi}$ is dense in $L_{\mathbf{0}}^{\boldsymbol{\psi}}$. ∎

Claim 4: $Z = \{x \in C : \boldsymbol{\psi}(x) \gg \mathbf{0}\}$.

Proof. The case $Z \subset \{x \in C : \boldsymbol{\psi}(x) \gg \mathbf{0}\}$ follows immediately from $\boldsymbol{\varphi} \gg \mathbf{0}$. Let $x \in C$ be such that $\boldsymbol{\psi}(x) \gg \mathbf{0}$. Select a $\mathbf{q} \in \mathbb{Q}^\omega$ such that $\mathbf{0} \ll \mathbf{q} \ll \boldsymbol{\psi}(x)$. According to Claim 2 we have $x \in \overline{\bigcap_{i=0}^n A_i(q_i)}$ for $n \in \omega$. We will find by recursion sets $U_n \in \mathfrak{U}_n$ such that for $n \in \omega$,

$$x \in \overline{\bigcap_{i=0}^{k} A_i(q_i) \cap \bigcap_{j=0}^{n} U_j} \quad \text{for each } k \in \omega. \tag{5.7}$$

For the base case note that since $\varphi_0 \leq \chi_0$, we have that the element $A_0(q_0)$ of $\mathcal{F}$ can be covered using only finitely many elements of $\mathfrak{U}_0$. It is easily seen that there must be a $U_0 \in \mathfrak{U}_0$ such that $x \in \overline{\bigcap_{i=0}^k A_i(q_i) \cap U_0}$ for each $k \in \omega$. The induction step is analogous to the base step.

The collection $\{U_i : i \in \omega\}$ clearly has the finite intersection property. Thus by assumption (4) of the theorem the sequence $(\bigcap_{i=0}^n U_i)_{n\in\omega}$ converges to a point in Z. That means that $\bigcap_{n\in\omega} \overline{\bigcap_{i=0}^n U_i}$ contains only one point which must be x, thus $x \in Z$. ∎

We now have that X is homeomorphic to $G_{\mathbf{0}}^{\boldsymbol{\varphi}} = G_{\mathbf{0}}^{\boldsymbol{\psi}}$ and that $G_{\mathbf{0}}^{\boldsymbol{\psi}}$ is homeomorphic to $\mathfrak{E}_{\mathrm{c}}^\omega$ by Theorem 4.3. ∎

THEOREM 5.4: *A nonempty space X is homeomorphic to $\mathfrak{E}_c^\omega$ if and only if there exist topologies $\mathcal{W}, \mathcal{T}_0, \mathcal{T}_1, \dots$ on X and covers $\mathcal{U}_i \subset \mathcal{T}_i$ of X for $i \in \omega$ such that*

(1) *the given topology $\mathcal{S}$ on X is equal to $\bigvee_{i\in\omega} \mathcal{T}_i$;*
(2′) *$Z = (X, \mathcal{W})$ is zero-dimensional and $\mathcal{W} \subset \mathcal{T}_i$ for each $i \in \omega$;*
(3′) *for each $i \in \omega$, $x \in X$, and neighbourhood U of x in $X_i = (X, \mathcal{T}_i)$ there is a neighbourhood V of x in X_i such that V is a nowhere dense subset of $(U, \bigvee_{j\neq i} \mathcal{T}_j)$ that is closed in Z; and*
(4) *if $U_i \in \mathcal{U}_i$ is such that $\{U_i : i \in \omega\}$ has the finite intersection property, then the sequence $(\bigcap_{j=0}^{i} U_j)_{i\in\omega}$ converges to a point in Z.*

Proof. We derive this theorem from Theorem 5.3.

(2)&(3) ⇒ (2′)&(3′). Assume that X satisfies (2) and (3) of Theorem 5.3, that $i \in \omega$, and that U is an open neighbourhood of some point x in X_i. Select with assumption (3) a neighbourhood W of x in X_i such that W contains no nonempty clopen subsets of any element of $\bigvee_{j\neq i} \mathcal{T}_j$. Select with (2) a neighbourhood V of x in E such that $V \subset U \cap W$ and V is closed in Z. Suppose that $O \in \bigvee_{j\neq i} \mathcal{T}_j$ is such that $O \cap U \subset V$. Note that $O \cap U$ is $\mathcal{T}_i$-open in O. On the other hand, $O \cap U = O \cap V$ is closed in $(O, \mathcal{W})$ and therefore also closed in $(O, \mathcal{T}_i)$ by (2). Thus we have that V and W contain the clopen subset $O \cap U$ of $(O, \mathcal{T}_i)$ and hence $O \cap U = \emptyset$. We have shown that V has an empty interior in $(U, \bigvee_{j\neq i} \mathcal{T}_j)$ and thus the set is nowhere dense because V is closed with respect to $\mathcal{W}$. We have proved property (3′). Property (2′) follows trivially from (2).

(2′)&(3′)&(4) ⇒ (2)&(3). Assume (2′), (3′), and (4) and note that (2) follows immediately from (2′) and (3′). Let x be a point in X and let $i \in \omega$. Put $\mathcal{T} = \bigvee_{j\neq i} \mathcal{T}_j$. By symmetry we may assume that $i = 0$. Let $U_0 \in \mathcal{U}_0$ be such that $x \in U_0$ and select with (3′) a neighbourhood $C_0 \subset U_0$ of x in X_0 that is closed in Z. Suppose that there is an $O \in \mathcal{T}$ and a clopen nonempty subset C of $(O, \mathcal{T}_0)$ such that $C \subset C_0$. Let $x' \in C$. Since Z is a witness to $(X, \mathcal{T})$ we can find a neighbourhood W of x' in $(X, \mathcal{T})$ such that $W \subset O$, W is closed in Z, and W is contained in a U_1 from $\mathcal{U}_1$. Put $C' = C \cap W$ and note that C' is clopen in $(W, \mathcal{T}_0)$.

With property (3′) choose for each $y \in C'$ a neighbourhood $V(y)$ of y in X_0 such that $V(y)$ is a nowhere dense subset of $C' \cup (X \setminus W)$ with the topology $\mathcal{T}$ and $V(y)$ is closed in Z. Since X_0 is separable metric we can find a countable set $\{a_j : j \in \mathbb{N}\} \subset C'$ with $C' \subset \bigcup\{V(a_j) : j \in \mathbb{N}\}$. Put $F_j = V(a_j) \cap W$ and note that $C' = \bigcup_{j=1}^{\infty} F_j$ with every F_j nowhere dense in $C' \cup (X \setminus W)$ with the topology $\mathcal{T}$ and closed in Z. Similarly, we can write $W \setminus C' = \bigcup_{j=1}^{\infty} G_j$ where every G_j is closed in Z.

We now construct recursively a sequence $C_1 \supset C_2 \supset \cdots$ of closed subsets of Z such that for every $n \in \mathbb{N}$,

(a) C_n is contained in an element U_n of $\mathcal{U}_n$;
(b) $C_n \cap (F_{n-1} \cup G_{n-1}) = \emptyset$; and
(c) C' meets the $\mathcal{T}$-interior of C_n.

For the basis step we put $C_1 = W$ and $F_0 = G_0 = \emptyset$ and we note that the hypotheses are satisfied (x' is both in C' and the $\mathcal{T}$-interior of W).

Assume that C_n has been found. Let P denote the $\mathcal{T}$-interior of C_n. Let $b \in C' \cap P$ and select a $U_{n+1} \in \mathcal{U}_{n+1}$ such that $b \in U_{n+1}$. Then $P \cap U_{n+1} \setminus G_n$ is an element of $\mathcal{T}$ that contains b. Since F_n is nowhere dense in $C' \cup (X \setminus W)$ with the topology $\mathcal{T}$ and $P \subset W$, we have that $P' = P \cap U_{n+1} \setminus (F_n \cup G_n) \in \mathcal{T}$ contains some point $c \in C'$. Select now a $\mathcal{T}$-neighbourhood C_{n+1} of c in P' that is closed in Z. The induction hypotheses are clearly satisfied for C_{n+1}.

Since $C' \subset C_0 \subset U_0$ we have that $C_0 \cap C_n$ is nonempty and contained in $\bigcap_{j=0}^{n} U_j$. Thus $\{U_j : j \in \omega\}$ has the finite intersection property and with condition (4) we have that the sequence $(\bigcap_{j=0}^{n} U_j)_{n \in \omega}$ converges to a point z in Z. Since the C_n are closed in Z we have $z \in \bigcap_{n \in \omega} C_n$. Since $C_1 = W$ we have $z \in W$. On the other hand, we have that hypothesis (b) and the fact $W = \bigcup_{n \in \mathbb{N}} F_n \cup G_n$ imply that $W \cap \bigcap_{n \in \omega} C_n = \emptyset$. We have proved condition (3). ∎

Let us examine the witness topologies on $\mathfrak{E}_{\mathrm{c}}^{\omega}$. We need some definitions.

If A is a nonempty set, then $A^{<\omega}$ denotes the set of all finite strings of elements of A, including the null string λ. Let A^{ω} denote the set of all infinite strings of elements of A. If $s \in A^{<\omega}$ and $\sigma \in A^{<\omega} \cup A^{\omega}$, then we put $s \prec \sigma$ if s is an initial substring of σ. If $\sigma \in A^{<\omega} \cup A^{\omega}$ and $k \in \omega$, then $\sigma \restriction k \in A^{<\omega}$ is the string of length k with $\sigma \restriction k \prec \sigma$. A **tree** T over a set A is a subset of $A^{<\omega}$ that is closed under initial segments, that is, if $s \in T$ and $t \prec s$, then $t \in T$. An **infinite branch** of T is an element σ of A^{ω} such that $\sigma \restriction k \in T$ for every $k \in \omega$. The **body** of T, written as $[T]$, is the set of all infinite branches of T. If $s \in T$ then $\mathrm{succ}(s)$ denotes the set of immediate successors of s in T.

Definition 5.5: Let X be a space. We call a system $(X_s)_{s\in T}$ of closed subsets of X a **generalized Sierpiński stratification** of X if the index set T is a tree as above such that:

i. $X_\lambda = X$ and $X_s = \bigcup\{X_t : t \in \operatorname{succ}(s)\}$ for all $s \in T$, and
ii. if $\sigma \in [T]$ then the sequence $X_{\sigma\restriction 0}, X_{\sigma\restriction 1}, \dots$ converges to a point $x_\sigma \in X$.

Sierpiński [24] has shown that X is an (absolute) $F_{\sigma\delta}$-space if and only if it admits such a stratification with a countable tree.

PROPOSITION 5.6: *If a nonempty space X admits a generalized Sierpiński stratification that consists of cohesive sets and if $\mathcal{W}$ is a topology on X that witnesses the almost zero-dimensionality of X, then $\emptyset$ is the only open subset of X that is an absolute $G_{\delta\sigma}$-subspace of $(X,\mathcal{W})$. If, moreover, $\mathcal{W}$ is an $F_{\sigma\delta}$-topology, then $(X,\mathcal{W})$ is homeomorphic to $\mathbb{Q}^\omega$.*

Proof. Let $(X_s)_{s\in T}$ be the generalized Sierpiński stratification. We may assume that every X_s is nonempty. Let O be a nonempty open subset of X and let $O = \bigcup_{i=1}^\infty G_i$, where every G_i is a topologically complete subspace of $Z = (X,\mathcal{W})$. Put $G_0 = \emptyset$. We construct by recursion a sequence $s_0 \prec s_1 \prec \cdots$ in T such that, for each $i \in \omega$, $|s_i| \geq i$ and $X_{s_i} \subset O \setminus G_i$. Let $a \in O$ and note that there is a $\tau \in [T]$ such that $a = x_\tau$ as in Definition 5.5. Since O is open in X and $X_{\tau\restriction 0}, X_{\tau\restriction 1}, \dots$ converges to x_τ, we have that there is an $s_0 \in T$ with $X_{s_0} \subset O = O \setminus G_0$. Now assume that s_i has been found. Since X_{s_i} is a G_δ-subset of Z by Remark 2.2, we have that $G_{i+1} \cap X_{s_i}$ is a topologically complete subspace of Z just like G_{i+1}. According to [12, Remark 5.5] we have by cohesion that $(X_{s_i}, \mathcal{W})$ is a first category space. Since $G_{i+1} \cap X_{s_i}$ is complete with respect to the topology $\mathcal{W}$ it cannot be dense in $(X_{s_i}, \mathcal{W})$. Thus there is an $x \in X_{s_i}$ and an open subset U of $(X_{s_i}, \mathcal{W})$ such that $x \in U \subset X_{s_i} \setminus G_{i+1}$. Note that U is also open in X_{s_i}. By the same method as employed above we can find an $s_{i+1} \in T$ such that $s_i \prec s_{i+1}$, $|s_{i+1}| \geq i+1$, and $X_{s_{i+1}} \subset U \subset O \setminus G_{i+1}$. This completes the recursion. There obviously exists a $\sigma \in [T]$ such that $s_i \prec \sigma$ for all i. Then $x_\sigma \in \bigcap_{i=0}^\infty X_{s_i} \subset O \setminus \bigcup_{i=1}^\infty G_i$. Since the G_i cover O we have a contradiction.

According to [12, Remark 5.5] $(X,\mathcal{W})$ is a first category space because $X = X_\lambda$ is cohesive. The second part of the proposition follows if we recall that a space is homeomorphic to $\mathbb{Q}^\omega$ if and only if it is a zero-dimensional, first category $F_{\sigma\delta}$-space with the property that no nonempty open subset is a $G_{\delta\sigma}$-space; see

Steel [25] or van Engelen [17, Theorem A.2.5]. ∎

Remark 5.7: In view of Theorem 4.2 and Claim 4 of the proof of Theorem 5.3, we have that the homeomorphism that links X with $\mathfrak{E}_c^\omega$ in Theorems 5.3 and 5.4 is also a homeomorphism on the level of the respective witness topologies. At first glance, this seems to restrict the applicability of these theorems. However, the following result shows that all witness topologies on $\mathfrak{E}_c^\omega$ are homeomorphic.

Corollary 5.8: *Let $Y_0, Y_1, \ldots$ be a sequence of nonempty, almost zero-dimensional complete spaces. If infinitely many of the Y_i are cohesive, then for every witness topology $\mathcal{W}$ for $X = \prod_{i\in\omega} Y_i$ we have that $(X, \mathcal{W})$ is homeomorphic to $\mathbb{Q}^\omega$.*

Proof. Let $T = \bigcup_{k=0}^\infty \prod_{i=0}^{k-1} Y_i$ and define the cohesive space $X_s = \{s\} \times \prod_{i=|s|}^\infty Y_i$ for $s \in T$. Clearly, if $\sigma \in [T] = X$ then $x_\sigma = \sigma$ and $(X_s)_{s\in T}$ is a generalized Sierpiński stratification of X. Thus Proposition 5.6 applies to X. According to [12, Remark 4.12] we have that $\mathcal{W}$ is an $F_{\sigma\delta}$-topology. ∎

Remark 5.9: Since the standard witness topology on $\mathfrak{E}_c$ is σ-compact, we have here an alternative proof of the fact that $\mathfrak{E}_c$ is not homeomorphic to $\mathfrak{E}_c^\omega$.

6. Applications

A subset A of a space X is called **negligible** if $X \setminus A$ is homeomorphic to X. Recall that the σ-compact subsets of $\mathbb{P}$ and ℓ^2 are negligible.

Corollary 6.1: *All σ-compacta in $\mathfrak{E}_c^\omega$ are negligible.*

Proof. Let $\mathcal{W}, \mathcal{T}_0, \mathcal{T}_1, \ldots$ be topologies and let $\mathcal{U}_i \subset \mathcal{T}_i$ be covers for $\mathfrak{E}_c^\omega$ as in Theorem 5.4. Let $K = \bigcup_{i\in\omega} K_i$ be a subset of $\mathfrak{E}_c^\omega$ such that each K_i is compact. Since every σ-compact almost zero-dimensional space is zero-dimensional, we have that K has an empty interior and, in particular, that $\mathfrak{E}_c^\omega \setminus K \neq \emptyset$.

Define $\mathcal{W}' = \{U \setminus K : U \in \mathcal{W}\}$ and $\mathcal{T}_i' = \{U \setminus K : U \in \mathcal{T}_i\}$ for $i \in \omega$. Clearly, conditions (1) and (2′) of Theorem 5.4 are satisfied for $\mathfrak{E}_c^\omega \setminus K$. For condition (3′) let $x \in \mathfrak{E}_c^\omega \setminus K$ and let $U \in \mathcal{T}_i$ be such that $x \in U$. Then we can find a $\mathcal{T}_i$-neighbourhood V of x that is $\mathcal{W}$-closed and that is a nowhere dense subset of $(U, \bigvee_{j\neq i} \mathcal{T}_j)$. Consider now the $\mathcal{W}'$-closed set $V' = V \setminus K$. Let O be an element of $\bigvee_{j\neq i} \mathcal{T}_j$ such that $O \cap U \setminus K \subset V$. Then $O \cap U \setminus V$ is an open subset of $\mathfrak{E}_c^\omega$ that is contained in K. Thus $O \cap U \setminus V = \emptyset$ and $O \cap U \subset V$, which means by the way V was chosen that $O \cap U = \emptyset$. We have shown that $V \setminus K$ is a nowhere dense subset of $(U \setminus K, \bigvee_{j \neq i} \mathcal{T}_j)$ and that (3′) is satisfied.

We now consider condition (4). Note that each K_i is also compact in $Z = (\mathfrak{E}_c^\omega, \mathcal{W})$ and hence we can write $\mathfrak{E}_c^\omega \setminus K_i = \bigcup_{j \in \omega} C_{ij}$ with each C_{ij} clopen in Z. Define

$$\mathcal{U}_i' = \{U \cap C_{ij} \setminus K : U \in \mathcal{U}_i,\, j \in \omega\} \subset \mathcal{T}_i' \tag{6.1}$$

for $i \in \omega$ and note that $\mathcal{U}_i'$ covers $\mathfrak{E}_c^\omega \setminus K$. Let $U_i' \in \mathcal{U}_i'$ be such that $\{U_i' : i \in \omega\}$ has the finite intersection property. Select for each $i \in \omega$ a $U_i \in \mathcal{U}_i$ such that $U_i' = U_i \cap C_{ij_i} \setminus K$ for some $j_i \in \omega$. Then also $\{U_i : i \in \omega\}$ has the finite intersection property, thus $(\bigcap_{k=0}^i U_k)_{i \in \omega}$ converges to a point x in Z. Then also $(\bigcap_{k=0}^i U_k')_{i \in \omega}$ converges to x in Z. Since C_{ij_i} is closed in Z and $U_i' \subset C_{ij_i}$, we have that $x \in C_{ij_i}$ for each $i \in \omega$. We may conclude that $x \notin K$ and hence $(\bigcap_{k=0}^i U_k')_{i \in \omega}$ converges to x in $Z \setminus K$. ∎

Remark 6.2: Note that we do not fully use the compactness of the K_i in the proof of Corollary 6.1, just that every K_i is nowhere dense in $\mathfrak{E}_c^\omega$ and $\mathcal{W}$-closed. So we have the following stronger result. Let $\mathcal{W}$ be a witness topology for $\mathfrak{E}_c^\omega$ that satisfies the conditions of Theorem 5.4. If K is a first category subset of $\mathfrak{E}_c^\omega$ that is an F_σ-set with respect to $\mathcal{W}$ then K is negligible in $\mathfrak{E}_c^\omega$.

σ-Compacta are also negligible in Erdős space and complete Erdős space; see Dijkstra and van Mill [14] and Kawamura, Oversteegen and Tymchatyn [19]. In [12, Corollary 8.15] and [19] it is also shown that closed proper subsets of $\mathfrak{E}$ and $\mathfrak{E}_c$ are negligible. This leads to

Question 6.3: Are all nonempty open subsets of $\mathfrak{E}_c^\omega$ homeomorphic to $\mathfrak{E}_c^\omega$?

Definition 6.4: A space X is called an **$\mathfrak{E}_c^\omega$-factor** if there is a space Y such that $X \times Y$ is homeomorphic to $\mathfrak{E}_c^\omega$.

We have the following stability theorem, which shows that $\mathfrak{E}_c^\omega$ is the 'maximal space' in the class of complete almost zero-dimensional spaces.

THEOREM 6.5: *For a nonempty space X the following statements are equivalent:*

(1) $X \times \mathfrak{E}_c^\omega \approx \mathfrak{E}_c^\omega$,
(2) *X is an $\mathfrak{E}_c^\omega$-factor,*
(3) *X is homeomorphic to a retract of $\mathfrak{E}_c^\omega$,*

(4) *X admits an imbedding as a C-set in $\mathfrak{E}_{\mathrm{c}}^{\omega}$,*
(5) *X admits a closed imbedding in $\mathfrak{E}_{\mathrm{c}}^{\omega}$,*
(6) *X is homeomorphic to a G_δ-subset of $\mathfrak{E}_{\mathrm{c}}^{\omega}$,*
(7) *there is an ω-USC function $\boldsymbol{\chi}\colon C \to \mathbb{I}^\omega$ such that C is complete,* $\dim C = 0$, *and $X \approx G_{\mathbf{0}}^{\boldsymbol{\chi}}$,*
(8) *there is an ω-USC function $\boldsymbol{\chi}\colon C \to \mathbb{I}^\omega$ such that C is compact,* $\dim C = 0$, *and $X \approx G_{\mathbf{0}}^{\boldsymbol{\chi}}$, and*
(9) *X is almost zero-dimensional and complete.*

Proof. (1) ⇒ (2), (2) ⇒ (3)&(4), (3) ⇒ (5), (4) ⇒ (5), (5) ⇒ (6), (6) ⇒ (9), and (8) ⇒ (7) are trivial. (7) ⇒ (9) follows from Remark 2.5.

(9) ⇒ (8)&(1). Assume (9) and choose a witness topology $\mathcal{W}$ for X. Let C be a zero-dimensional compactification of $(X, \mathcal{W})$ and apply Lemma 3.5 to obtain (8) with the additional property that $G_{\mathbf{0}}^{\boldsymbol{\chi}}$ is dense in the graph of $\boldsymbol{\chi}$. Let $\boldsymbol{\psi}$ be an ω-Lelek function such that $G_{\mathbf{0}}^{\boldsymbol{\psi}} \approx \mathfrak{E}_{\mathrm{c}}^{\omega}$ as described in Remark 2.8. According to Lemma 3.4 we have that $\mathfrak{E}_{\mathrm{c}}^{\omega} \times X \approx G_{\mathbf{0}}^{\boldsymbol{\psi}\times\boldsymbol{\chi}}$ and that $\boldsymbol{\psi}\times\boldsymbol{\chi}$ is ω-Lelek. Apply now Theorem 4.3 to obtain $G_{\mathbf{0}}^{\boldsymbol{\psi}\times\boldsymbol{\chi}} \approx \mathfrak{E}_{\mathrm{c}}^{\omega}$. ∎

Remark 6.6: To prove the implication (9) ⇒ (1) we used the extrinsic characterization Theorem 4.3 in combination with Lemmas 3.5 and 3.4 because this method also gives us condition (8). One can derive (9) ⇒ (1) directly from the intrinsic characterizations as follows. Select $\mathcal{W}, \mathcal{T}_0, \mathcal{T}_1, \ldots$ and $\mathcal{U}_0, \mathcal{U}_1, \ldots$ for $\mathfrak{E}_{\mathrm{c}}^{\omega}$ as in Theorem 5.4, for instance. Select a complete metric d for X and a witness topology $\mathcal{W}'$ for X. Let $\mathcal{T}_i'$ simply be the given topology on X for every $i \in \omega$ and define, for $i \in \omega$,

$$\mathcal{U}_i' = \{O \in \mathcal{T}_i' : \operatorname{diam} O < 2^{-i}\}. \tag{6.2}$$

If we put $(X \times \mathfrak{E}_{\mathrm{c}}^{\omega}, \mathcal{W}'') = (X, \mathcal{W}') \times (\mathfrak{E}_{\mathrm{c}}^{\omega}, \mathcal{W})$, $(X \times \mathfrak{E}_{\mathrm{c}}^{\omega}, \mathcal{T}_i'') = (X, \mathcal{T}_i') \times (\mathfrak{E}_{\mathrm{c}}^{\omega}, \mathcal{T}_i)$, and $\mathcal{U}_i'' = \{O' \times O : O' \in \mathcal{U}_i',\ O \in \mathcal{U}_i\}$, then it is easily verified that the premises of Theorem 5.4 are satisfied for $X \times \mathfrak{E}_{\mathrm{c}}^{\omega}$.

Remark 6.7: If $X \times Y \approx Z^\omega$, then $Z^\omega \approx (Z^\omega)^\omega \approx (X \times Y)^\omega \approx X \times (X \times Y)^\omega \approx X \times Z^\omega$. Thus conditions (1) and (2) as in Theorem 6.5 are always equivalent for spaces that are infinite powers.

Since every totally disconnected σ-compactum is zero-dimensional, the stable space for the (σ-)compact almost zero-dimensional spaces is 2^ω ($2^\omega \times \mathbb{Q}$).

We finish by showing that $\mathfrak{E}_{\mathrm{c}}^{\omega}$ is countable dense homogeneous just like $\mathbb{R}$, 2^{ω}, $\mathbb{P}$, Q, and ℓ^2. This is an application not of the characterization theorems but of a theorem in Dijkstra [9].

Definition 6.8: A space X is **countable dense homogeneous** (**CDH**) if, given any two countable dense subsets $A, B \subset X$, there is a homeomorphism h of X such that $h(A) = B$. A space X is called **strongly locally homogeneous** (**SLH**) if there is a basis $\mathcal{B}$ for the topology such that, for every $B \in \mathcal{B}$ and $x, y \in B$, there exists a homeomorphism $h\colon X \to X$ that is supported on B and that maps x to y. (A function $f\colon X \to X$ is said to be **supported on** a subset A of X if the restriction $f{\restriction}(X \setminus A)$ is the identity.)

The standard method for proving that a space is CDH uses a theorem of Bennett [3], which states that for complete spaces SLH implies CDH. This method does not work for $\mathfrak{E}_{\mathrm{c}}^{\omega}$ because that space is not SLH:

Proposition 6.9: *Hereditarily disconnected SLH spaces are zero-dimensional.*

Recall that a space is called **hereditarily disconnected** if every component is a singleton.

Proof. Let X be an SLH space that is hereditarily disconnected and let $\mathcal{B}$ be a basis as in Definition 6.8. Let $B \in \mathcal{B}$ and $x \in B$. It suffices to show that there is a clopen set O in X with $x \in O \subset B$. If $B = \{x\}$, then x is isolated and we are finished. So assume that B is not a singleton. Then by hereditary disconnectedness there is a clopen subset C of $\overline{B}$ such that $x \in C$ and $C \neq \overline{B}$ (and hence $B \setminus C \neq \emptyset$). Let $h\colon X \to X$ be a homeomorphism that is supported on B such that $h(x) \in B \setminus C$. Define the closed set $O = C \cap h^{-1}(\overline{B} \setminus C)$ and note that $x \in O$. If $y \in O$, then $h(y) \neq y$ so $y \in B$ and $h(y) \in B$. This means that $y \in (B \cap C) \cap h^{-1}(B \setminus C)$. Thus O equals the clearly open set $(B \cap C) \cap h^{-1}(B \setminus C)$. We have that O is clopen in X and contained in B, thus X is zero-dimensional. ∎

Definition 6.10: Let (X, d) be a metric space and let $f : X \to X$ be a bijection. Then the **norm** of f is defined by

$$\|f\| = \sup\left\{ \left| \log \frac{d(f(x), f(y))}{d(x,y)} \right| : x, y \in X \text{ with } x \neq y \right\} \in [0, \infty]. \tag{6.3}$$

The space (X, d) is said to be **LSLH**$^-$ if for any $x \in X$, any $\varepsilon > 0$, and any finite subset F of $X \setminus \{x\}$ there is a neighbourhood V of x such that, for each $y \in V$, there is a permutation f of X such that points of F are fixed, $\|f\| \leq \varepsilon$, and $f(x) = y$.

The following theorem is from [9].

THEOREM 6.11: *If (X, d) is a complete metric space that is LSLH$^-$, then X is countable dense homogeneous.*

PROPOSITION 6.12: *The space $\mathfrak{E}_{\mathrm{c}}^{\omega}$ is countable dense homogeneous.*

Proof. We begin by presenting a particularly elegant model of $\mathfrak{E}_{\mathrm{c}}$ that is featured in Dijkstra [7] and called *harmonic Erdős space*. Consider the Cantor set $C = 2^\omega = \{0,1\}^\omega$ with its standard boolean group structure $\triangle$. We define the following 'norm' from C to $[0, \infty]$:

$$\varphi(x) = \sum_{n=0}^{\infty} \frac{x_n}{n+1}. \tag{6.4}$$

Note that $\varphi(x \triangle y) \leq \varphi(x) + \varphi(y)$ for all $x, y \in C$ and hence $\mathfrak{E}_{\mathrm{h}} = \{x \in C : \varphi(x) < \infty\}$ is a subgroup of C. Moreover, it follows that $d(x, y) = \varphi(x \triangle y)$ defines an invariant metric on $\mathfrak{E}_{\mathrm{h}}$ that makes $\mathfrak{E}_{\mathrm{h}}$ into a topological group. It is shown in [7] that $(\mathfrak{E}_{\mathrm{h}}, d)$ is homeomorphic to $\mathfrak{E}_{\mathrm{c}}$ and that d is complete. Consider now the space $\mathfrak{E}_{\mathrm{h}}^{\omega} = \prod_{i\in\omega} \mathfrak{E}_{\mathrm{h}}$ with the function $\psi\colon \mathfrak{E}_{\mathrm{h}}^{\omega} \to [0,1]$ that is given by

$$\psi(\mathbf{x}) = \max_{i\in\omega}(\min\{1/(i+1), \varphi(x_i)\}), \tag{6.5}$$

where $\mathbf{x} = (x_0, x_1, \dots) \in \mathfrak{E}_{\mathrm{h}}^{\omega}$. We define the natural boolean group structure on $\mathfrak{E}_{\mathrm{h}}^{\omega}$ by $\mathbf{x} \triangle \mathbf{y} = (x_0 \triangle y_0, x_1 \triangle y_1, \dots)$. Note that the invariant complete metric $\rho(\mathbf{x}, \mathbf{y}) = \psi(\mathbf{x} \triangle \mathbf{y})$ generates the product topology on $\mathfrak{E}_{\mathrm{h}}^{\omega}$. If $\mathbf{x} = (x_0, x_1, \dots) \in \mathfrak{E}_{\mathrm{h}}^{\omega}$, then each $x_i \in 2^\omega$ and we will use the notation $x_i = (x_{i0}, x_{i1}, \dots)$ where $x_{ij} \in \{0,1\}$.

To show that $(\mathfrak{E}_{\mathrm{h}}^{\omega}, \rho)$ is LSLH$^-$ and hence CDH by Theorem 6.11, let $\mathbf{a} \in \mathfrak{E}_{\mathrm{h}}^{\omega}$, let $F \subset \mathfrak{E}_{\mathrm{h}}^{\omega} \setminus \{\mathbf{a}\}$ be finite, and let $\varepsilon > 0$. Select an $n \in \mathbb{N}$ such that for each $\mathbf{b} \in F$ there exist $i, j < n$ with $b_{ij} \neq a_{ij}$. Put $\delta = \frac{1}{n}(1 - e^{-\varepsilon})$ and define the clopen neighbourhood $U = \{\mathbf{x} \in \mathfrak{E}_{\mathrm{h}}^{\omega} : x_{ij} = a_{ij} \text{ for } i, j < n\}$ of $\mathbf{a}$. Let $\mathbf{b} \in U$ be such that $\rho(\mathbf{a},\mathbf{b})<\delta$ and define the function $h\colon \mathfrak{E}_{\mathrm{h}}^{\omega}\to\mathfrak{E}_{\mathrm{h}}^{\omega}$ by

$$h(x)=\begin{cases}\mathbf{x}\bigtriangleup\mathbf{a}\bigtriangleup\mathbf{b}, & \text{if } \mathbf{x}\in U;\\ \mathbf{x}, & \text{if } \mathbf{x}\in\mathfrak{E}_{\mathrm{h}}^{\omega}\setminus U.\end{cases}\tag{6.6}$$

Note that h is supported on U, thus h fixes the points of F. Also, $h(\mathbf{a})=\mathbf{b}$ and $h\restriction U$ is an isometry of U because ρ is invariant. So to estimate $\|h\|$ we only have to consider points $\mathbf{x}\in U$ and $\mathbf{y}\in\mathfrak{E}_{\mathrm{h}}^{\omega}\setminus U$. Thus there are $i,j<n$ with $x_{ij}\neq y_{ij}$ and hence

$$\varphi(x_i\bigtriangleup y_i)\geq\frac{1}{j+1}\quad\text{and}\quad\rho(\mathbf{x},\mathbf{y})\geq\min\left\{\frac{1}{i+1},\frac{1}{j+1}\right\}\geq\frac{1}{n}.$$

Since $h(\mathbf{y})=\mathbf{y}$ we have

$$\left|\frac{\rho(h(\mathbf{x}),h(\mathbf{y}))}{\rho(\mathbf{x},\mathbf{y})}-1\right|\leq\frac{\rho(h(\mathbf{x}),\mathbf{x})}{\rho(\mathbf{x},\mathbf{y})}\leq\frac{\varphi(\mathbf{x}\bigtriangleup\mathbf{a}\bigtriangleup\mathbf{b}\bigtriangleup\mathbf{x})}{1/n}=n\psi(\mathbf{a}\bigtriangleup\mathbf{b})<n\delta.\tag{6.7}$$

Thus $\|h\|\leq-\log(1-n\delta)=\varepsilon$. ∎

Kawamura, Oversteegen and Tymchatyn [19] have shown that $\mathfrak{E}_{\mathrm{c}}$ is CDH.

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